Sets

Emily

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Introduction

This chapter is supposed to eventually contain stuff on axiomatic set theory, although currently it has very little material in this direction.

Section A also contains a discussion of negative thinking in the sense of [nLab23] and how sets may be viewed as categories enriched in truth values (see in particular Section A.5).

Notes to Myself

1. https://mathoverflow.net/questions/436346/zorns-lemma-old-friend-or-historical-relic
2. Improve the typography of the table of analogies between set theory and category theory.
3. von Neumann hierarchy and sets hereditarily of cardinality less than κ
4. Axiom of regularity consequences:
   (a) No set is an element of itself.
   (b) There exists no infinite sequence $(a_i)_{i\in I}$ such that $a_{i+1} \in a_i$ for each $i \in I$.
5. construction of real numbers via dedekind cuts
6. TODO: NBG set theory.
7. ETCS.
8. Ultrafilters as probability measures

Contents

1 Zermelo–Fraenkel Set Theory

1.1 Foundations

Definition 1.1.1 ▶ Zermelo–Fraenkel Set Theory

Zermelo–Fraenkel set theory (ZF) is the theory determined by the primitive notions of:

1. "Sets";
2. "Belonging" to a set;

together with the following axioms:[1,2]

1. *The Axiom of Set Existence.* There exists a set.[3]

2. *The Axiom of Extensionality.* Let X and Y be sets. The following conditions are equivalent:

   (a) We have $X = Y$.

   (b) The sets X and Y have the same elements.

3. *The Axiom of Regularity*[4]. Every set X contains an element Y such that $X \cap Y = \emptyset$.

4. *The Axiom Scheme of Restricted*[5] *Comprehension*[6]. For each

   (a) Set X;

   (b) Formula $\phi(x, w_1, \dots, w_n)$ with free variables $x, w_1, \dots, w_n$;

   the set[7]
   $$\{x \in X \mid \phi(x, w_1, \dots, w_n)\}$$
   exists.

5. *The Axiom of Pairing.* For each pair of sets X and Y, there exists a set Z containing both X and Y as elements.

6. *The Axiom of Union.* For each set $\mathcal{F} = \{A_i\}_{i \in I}$, there exists a set F containing each element of each of the elements of $\mathcal{F}$ (i.e. containing the union of the elements of $\mathcal{F}$).

7. *The Axiom Scheme of Replacement.* For each

   - Formula $\phi(s, t, U, w)$ with free variables s, t, U, and w;
   - Set X;
   - Parameter p;

   If $\phi(s, t, X, p)$ defines a function F on X by
   $$F(x) = y \iff \phi(x, y, A, p),$$
   then there exists a set Y with $\mathrm{Im}(f) \subset Y$.

8. *The Axiom of Infinity.* There exists a set X satisfying the following conditions:[8]

   - We have $\emptyset \in X$;

- If $x \in X$, then $\mathrm{succ}(x) \in X$.

9. *The Axiom of Powerset.* For each each set X, there exists a set P containing every subset of X as an element.

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[1]Or "axiom schemes", which comprise an infinite number of axioms.

[2]The axioms of

- Pairing;
- Union;
- Powerset;
- Replacement;

don't produce exactly

- The pairing $\{X, Y\}$ of two sets;
- The union of the sets in a family, nor;
- The powerset of a set;
- The image of a definable function f;

but only sets containing these as subsets. (So, for instance, all that the axiom of pairing guarantees is the existence of a set containing the sets X and Y as elements, but not necessarily *only these* as elements: it may very well have the form $Z = \{X, Y, \text{other stuff}\}$.)

To actually construct these operations, we need instead to combine the above axioms with the axiom scheme of restricted comprehension; see Constructions With Sets, Definitions 2.3.1, 2.4.1 and 3.2.1.

[3]The axiom of set existence follows from the axiom of infinity, and is hence often omitted in a number of presentations of the ZF axioms.

[4]*Further Terminology:* Also called the **axiom of foundation**.

[5]The "restricted" in the name "axiom scheme of restricted comprehension" refers to the fact that this axiom can only be used to construct subsets of already existing sets, having the form

$$\{x \in X \mid \phi(x, w_1, \dots, w_n)\}.$$

Naive set theory, on the other hand, has an "axiom scheme of unrestricted comprehension", which can build "sets" of the more general form

$$\{x \mid \phi(x, w_1, \dots, w_n)\}.$$

The problem with unrestricted comprehension, however, is that it gives rise to contradictions, such as Russell's paradox.

[6]*Further Terminology:* Also called the **axiom scheme of specification** or the **axiom scheme of separation**.

[7]For $\phi \colon X \to \{\mathrm{true}, \mathrm{false}\}$ a formula with one free variable x, this set is a decategorified form of the category of elements of a functor, being the pullback

$$\begin{aligned} \{x \in X \mid \phi(x)\} &\cong X \times_{\{\mathrm{true},\mathrm{false}\}} \{\mathrm{true}, \mathrm{false}\}_{\mathrm{true}/} \\ &\cong X \times_{\{\mathrm{true},\mathrm{false}\}} \mathrm{pt}, \end{aligned} \qquad \begin{array}{ccc} \{x \in X \mid \phi(x)\} & \longrightarrow & \overbrace{\{\mathrm{true}, \mathrm{false}\}_{\mathrm{true}/}}^{\cong \mathrm{pt}} \\ \big\downarrow & \lrcorner & \big\downarrow \text{忘} \\ X & \xrightarrow[\phi]{} & \{\mathrm{true}, \mathrm{false}\}, \end{array}$$

while the category of elements of a functor F is the pullback

$$\int_C F \cong C \times_{\mathsf{Sets}} \mathsf{Sets}_{\mathrm{pt}/}, \qquad \begin{array}{ccc} \int_C F & \longrightarrow & \mathsf{Sets}_{\mathrm{pt}/} \\ \big\downarrow & \lrcorner & \big\downarrow \text{忘} \\ C & \xrightarrow[F]{} & \mathsf{Sets}. \end{array}$$

[8] See ?? and Definition 1.4.2 for the definitions of the empty set and the succ operator.

Remark 1.1.2 ▶ Slogans for the ZF Axioms

The ZF axioms may be roughly summarised by the following slogans:

1. *The Axiom of Set Existence.* There exists a set.
2. *The Axiom of Extensionality.* A set is uniquely determined by its elements.
3. *The Axiom of Regularity.* A set contains a set disjoint to itself.
4. *The Axiom Scheme of Restricted Comprehension.* Given a set X and a formula ϕ, there is a subset U of X whose elements are the elements of X satisfying ϕ.
5. *The Axiom of Pairing.* Given two sets, there exists a set containing them both.
6. *The Axiom of Union.* Given a family of sets, there exists a set containing the elements of each set in this family.
7. *The Axiom Scheme of Replacement.* The image of a definable function exists.
8. *The Axiom of Infinity.* There exists an infinite set (of a certain form).
9. *The Axiom of Powerset.* Given a set X, there exists a set containing every subset of X.

Proposition 1.1.3 ▶ Elementary Consequences of the ZF Axioms

Let X be a set.

1. *Non-Existence of the Set of All Sets.* There is no set of all sets.

Proof 1.1.4 ▶ Proof of Proposition 1.1.3

Item 1: Non-Existence of the Set of All Sets

If there were a set of all sets S, then we would be able to construct the set from Russell's paradox by applying the axiom scheme of restricted comprehension (Item 4 of Definition 1.1.1) to the formula $x \notin x$:

$$R \overset{\text{def}}{=} \{X \in S \mid x \notin x\}.$$

This then leads to a contradiction as soon as one asks: "Does R contain itself?".

1.2 Functions

Let A and B be sets.

Definition 1.2.1 ▶ Functions

A **function from** A **to** B[1] is a relation $f\colon A \nrightarrow B$ from A to B such that if $a \sim_f b$ and $a \sim_f c$, then $b = c$.

[1] *Further Terminology:* Also called a **map of sets from** A **to** B.

Definition 1.2.2 ▶ The Category of Sets

The **category of sets** is the category Sets where

- *Objects.* The objects of Sets are sets;
- *Morphisms.* The morphisms of Sets are functions;
- *Identities.* For each $X \in \mathrm{Obj}(\mathsf{Sets})$, the unit map

  $$\mathbb{1}^{\mathsf{Sets}}_X\colon \mathrm{pt} \to \mathsf{Sets}(X, X)$$

  of Sets at X is defined by

  $$\mathrm{id}^{\mathsf{Sets}}_X \overset{\text{def}}{=} \mathrm{id}_X;$$

- *Composition.* For each $X, Y, Z \in \mathrm{Obj}(\mathsf{Sets})$, the composition map

  $$\circ^{\mathsf{Sets}}_{X,Y,Z}\colon \mathsf{Sets}(Y, Z) \times \mathsf{Sets}(X, Y) \to \mathsf{Sets}(X, Z)$$

  of Sets at (X, Y, Z) is defined by

  $$g \circ^{\mathsf{Sets}}_{X,Y,Z} f \overset{\text{def}}{=} g \circ f$$

  for each $f \in \mathsf{Sets}(X, Y)$ and each $g \in \mathsf{Sets}(Y, Z)$.

1.2.1 The Associated Inclusion of Characteristic Relations

Let $f\colon A \to B$ be a map of sets.

Definition 1.2.3 ▶ The Inclusion of Characteristic Relations Associated to a Function

The **inclusion of characteristic relations associated to** f is the inclusion of relations

$$\chi_B \circ (f \times f) \subset \chi_A, \qquad \begin{array}{ccc} A \times A & \xrightarrow{\chi_A(-_1,-_2)} & \{\text{true}, \text{false}\} \\ {\scriptstyle f\times f}\downarrow & \subset & \downarrow{\scriptstyle \text{id}_{\{\text{true},\text{false}\}}} \\ B \times B & \xrightarrow[\chi_B(-_1,-_2)]{} & \{\text{true}, \text{false}\}. \end{array}$$

Proof 1.2.4 ▶ Proof of Definition 1.2.3

We claim that we indeed have the stated inclusion of relations:

1. If $\chi_A(x, y) = \text{true}$, then $x = y$ and we have
$$\begin{aligned} \chi_B(f(x), f(y)) &= \chi_B(f(x), f(x)) \\ &= \text{true}, \end{aligned}$$
so that we have a morphism
$$\chi_A(x, y) \to \chi_B(f(x), f(y))$$
given by $\text{id}_{\text{true}}\colon \text{true} \to \text{true}$;
2. If $\chi_A(x, y) = \text{false}$, then:
   (a) If $f(x) \neq f(y)$, then $\chi_B(f(x), f(y)) = \text{false}$, and we have a morphism
   $$\chi_A(x, y) \to \chi_B(f(x), f(y))$$
   given by $\text{id}_{\text{false}}\colon \text{false} \to \text{false}$;
   (b) If $f(x) = f(y)$, then $\chi_B(f(x), f(y)) = \text{true}$, and we have a morphism
   $$\chi_A(x, y) \to \chi_B(f(x), f(y))$$
   given by $!\colon \text{false} \to \text{true}$.

Thus, we have $\chi_A \subset \chi_B \circ (f \times f)$. ∎

1.3 The Axiom of Choice

Definition 1.3.1 ▶ The Axiom of Choice

The **axiom of choice** is the following axiom:

($\star$) For each family $\{S_i\}_{i\in I}$ of nonempty sets, there exists an indexed set $\{x_i\}_{i\in I}$ such that, for each $i \in I$, we have $x_i \in S_i$.

Proposition 1.3.2 ▶ Equivalents of the Axiom of Choice

Let $\{S_i\}_{i\in I}$ be a family of sets. The following conditions are equivalent:

1. The axiom of choice is true.
2. If, for each $i \in I$, we have $S_i \neq \emptyset$, then $\prod_{i\in I} S_i \neq \emptyset$.
3. There exists a choice function for $\{S_i\}_{i\in I}$.

Proof 1.3.3 ▶ Proof of Proposition 1.3.2

Omitted.

1.4 The Set of Natural Numbers

Definition 1.4.1 ▶ The Axioms of Peano Arithmetic

The **axioms of Peano arithmetic**[1] are the following axioms:

1. For each $n \in \mathbb{N}$, we have $0 \neq \mathrm{succ}(n)$.
2. For each $n, m \in \mathbb{N}$, if $\mathrm{succ}(n) = \mathrm{succ}(m)$, then $n = m$.
3. If 0 has property P and, for each $n \in \mathbb{N}$, the condition

   ($\star$) If n has property P, then $\mathrm{succ}(n)$ has property P.

   is true, then n has property P for each $n \in \mathbb{N}$.

[1] *Further Terminology:* Also called the **Peano axioms**, the **Dedekind–Peano axioms**, or the **Peano postulates**.

1.4.1 Successors

Let A be a set.

Definition 1.4.2 ▶ The Successor of a Set

The **successor of** A is the set $\mathrm{succ}(A)$ defined by

$$\mathrm{succ}(A) \stackrel{\mathrm{def}}{=} A \cup \{A\}.$$

2 Other Set Theories

2.1 Von Neumann–Bernays–Gödel Set Theory

2.2 Quine's New Foundations

Appendices

A The Enrichment of Sets in Classical Truth Values

A.1 (-2)-Categories

Definition A.1.1 ▶ (-2)-Categories

A (-2)**-category** is the "necessarily true" truth value.[1,2,3]

[1] That is, there is only one (-2)-category: "necessarily true".

[2] A $(-n)$-category for $n = 3, 4, \ldots$ is also the "necessarily true" truth value, coinciding with a (-2)-category.

[3] For motivation, see [BB10, p. 13].

A.2 (-1)-Categories

A.2.1 Foundations

Definition A.2.1 ▶ (-1)-Categories

A (-1)**-category** is a classical truth value.

Remark A.2.2 ▶ Motivation for (-1)-Categories

[1] (-1)-categories should be thought of as being "categories enriched in (-2)-categories", having a collection of objects and, for each pair of objects, a Hom-object $\mathrm{Hom}(x, y)$ that is a (-2)-category (i.e. trivial).

Therefore, a (-1)-category C is either ([BB10, pp. 33–34]):

1. *Empty*, having no objects;
2. *Contractible*, having a collection of objects $\{a, b, c, \ldots\}$, but with $\mathrm{Hom}_C(a, b)$ being a (-2)-category (i.e. trivial) for all $a, b \in \mathrm{Obj}(C)$, forcing all objects of C to be uniquely isomorphic to each other.

As such, there are only two (-1)-categories, up to equivalence:

- The (-1)-category false (the empty one);
- The (-1)-category true (the contractible one).

[1] For more motivation, see [BB10, p. 13].

A.2.2 The Set of (-1)-Categories

Definition A.2.3 ▶ The Set of (-1)-Categories

The **set of (-1)-categories** is the set Cats_{-1} defined by

$$\mathrm{Cats}_{-1} \overset{\text{def}}{=} \{\mathrm{true}, \mathrm{false}\}.$$

Remark A.2.4 ▶ Algebraic Structures on Cats_{-1}

The set Cats_{-1} admits a number of algebraic structures, each carrying a different name:

- *The Field With One Element.* The field with one element $\mathbb{F}_1$ of Monoids With Zero, Example 1.1.4;
- *The Boolean Monoid.* The Boolean monoid $\mathbb{B}$ of Monoids, Example 1.1.4;
- *The Group of Integers Modulo 2.* The group $\mathbb{Z}_{/2}$ of Groups, Definition 3.1.4;
- *The Boolean Semiring.* The Boolean semiring $\mathbb{B}$ of Commutative Semirings, **??**;

- *The Ring of Integers Modulo* 2. The ring $\mathbb{Z}_{/2}$ of Commutative Rings, Example 1.1.5.

Among these, it is the algebraic structure of the field with one element that comes into play when trying to express sets as being enriched over $\{\mathrm{true}, \mathrm{false}\}$: indeed, we have an isomorphism of categories[1]

$$\mathsf{Sets} \cong \mathsf{Cats}_{\mathbb{F}_1^{\mathrm{disc}}}.$$

[1]A small subtlety here is that we need to allow categories to be enriched not only over monoidal categories, but also over monoidal categories with zero.

A.2.3 The Poset of (-1)-Categories

Definition A.2.5 ▶ The Poset of (-1)-Categories

The **poset of (-1)-categories** is the poset $\left(\mathsf{Cats}^{\mathsf{Pos}}_{-1}, \preceq\right)$ consisting of[1]

- *The Underlying Set.* The set Cats_{-1} of Definition A.2.3;
- *The Partial Order.* The partial order

$$\preceq \colon \mathsf{Cats}_{-1} \times \mathsf{Cats}_{-1} \longrightarrow \underbrace{\{\mathrm{true}, \mathrm{false}\}}_{\stackrel{\mathrm{def}}{=}\mathsf{Cats}_{-1}}$$

on Cats_{-1} defined by

$$\begin{aligned} \mathrm{false} \preceq \mathrm{false} &\stackrel{\mathrm{def}}{=} \mathrm{true},\\ \mathrm{true} \preceq \mathrm{false} &\stackrel{\mathrm{def}}{=} \mathrm{false},\\ \mathrm{false} \preceq \mathrm{true} &\stackrel{\mathrm{def}}{=} \mathrm{true},\\ \mathrm{true} \preceq \mathrm{true} &\stackrel{\mathrm{def}}{=} \mathrm{true}. \end{aligned}$$

[1]As a posetal category, $\mathsf{Cats}^{\mathsf{Pos}}_{-1}$ is the category $\mathsf{Cats}^{\mathsf{cat}}_{-1}$ where

- We have

$$\begin{aligned} \mathrm{Obj}(\mathsf{Cats}^{\mathsf{cat}}_{-1}) &\stackrel{\mathrm{def}}{=} \mathsf{Cats}_{-1}\\ &\stackrel{\mathrm{def}}{=} \{\mathrm{true}, \mathrm{false}\}; \end{aligned}$$

- *Morphisms.* We have

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Cats}^{\mathsf{cat}}_{-1}}(\mathrm{true}, \mathrm{true}) &\stackrel{\mathrm{def}}{=} \{\mathrm{id}_{\mathrm{true}}\} \cong \mathrm{pt},\\ \mathrm{Hom}_{\mathsf{Cats}^{\mathsf{cat}}_{-1}}(\mathrm{true}, \mathrm{false}) &\stackrel{\mathrm{def}}{=} \emptyset \stackrel{\mathrm{def}}{=} \emptyset,\\ \mathrm{Hom}_{\mathsf{Cats}^{\mathsf{cat}}_{-1}}(\mathrm{false}, \mathrm{true}) &\stackrel{\mathrm{def}}{=} \{!\} \cong \mathrm{pt},\\ \mathrm{Hom}_{\mathsf{Cats}^{\mathsf{cat}}_{-1}}(\mathrm{false}, \mathrm{false}) &\stackrel{\mathrm{def}}{=} \{\mathrm{id}_{\mathrm{false}}\} \cong \mathrm{pt}; \end{aligned}$$

- *Identities.* The two unit maps

$$\mathbb{1}^{\mathsf{Cats}^{\mathsf{cat}}_{-1}}_{\mathsf{true}} \colon \mathsf{pt} \longrightarrow \mathrm{Hom}_{\mathsf{Cats}^{\mathsf{cat}}_{-1}}(\mathsf{true}, \mathsf{true}),$$
$$\mathbb{1}^{\mathsf{Cats}^{\mathsf{cat}}_{-1}}_{\mathsf{false}} \colon \mathsf{pt} \longrightarrow \mathrm{Hom}_{\mathsf{Cats}^{\mathsf{cat}}_{-1}}(\mathsf{false}, \mathsf{false})$$

  of $\mathsf{Cats}^{\mathsf{cat}}_{-1}$ are defined by

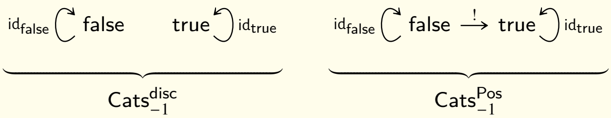
$$\mathrm{id}^{\mathsf{Cats}^{\mathsf{cat}}_{-1}}_{\mathsf{true}} \overset{\text{def}}{=} \mathrm{id}_{\mathsf{true}},$$
$$\mathrm{id}^{\mathsf{Cats}^{\mathsf{cat}}_{-1}}_{\mathsf{false}} \overset{\text{def}}{=} \mathrm{id}_{\mathsf{false}};$$

- *Composition.* The composition maps of $\mathsf{Cats}^{\mathsf{cat}}_{-1}$ are completely determined by the axioms for it to be a category.

Thus, the sole difference between $\mathsf{Cats}^{\mathsf{cat}}_{-1}$ and $\mathsf{Cats}^{\mathsf{disc}}_{-1}$ is that in $\mathsf{Cats}^{\mathsf{cat}}_{-1}$ we have a morphism $\mathsf{false} \longrightarrow \mathsf{true}$, whereas in $\mathsf{Cats}^{\mathsf{disc}}_{-1}$ we have none:

$$\mathrm{Hom}_{\mathsf{Cats}^{\mathsf{cat}}_{-1}}(\mathsf{false}, \mathsf{true}) \overset{\text{def}}{=} \mathsf{pt},$$
$$\mathrm{Hom}_{\mathsf{Cats}^{\mathsf{disc}}_{-1}}(\mathsf{false}, \mathsf{true}) \overset{\text{def}}{=} \emptyset.$$

Here is a picture of $\mathsf{Cats}^{\mathsf{disc}}_{-1}$ vs. $\mathsf{Cats}^{\mathsf{cat}}_{-1}$:



Remark A.2.6 ▶ Monoidal Structures on $\mathsf{Cats}^{\mathsf{cat}}_{-1}$

The category $\mathsf{Cats}^{\mathsf{cat}}_{-1}$ admits two symmetric monoidal category structures:

1. We have the Cartesian monoidal structure, whose tensor product is given by the categorical product, which is given on objects by[1]

$\times$	false	true
false	false	false
true	false	true

   This monoidal structure is both symmetric and closed, with internal **Hom** being given by

$\mathbf{Hom}_{\{t,f\}}(-_1, -_2)$	false	true
false	true	true
true	false	true

2. We have the coCartesian monoidal structure, whose tensor product is given by the categorical coproduct, which is given on objects by[2]

$\coprod$	false	true
false	false	true
true	true	true

This monoidal structure is symmetric, but it isn't closed.

[1] *Further Notation:* We also write $\wedge$ (read "and") for $\times$.
[2] *Further Notation:* We also write $\vee$ (read "or") for $\coprod$.

Remark A.2.7 ▶ The Monoidal Category With Zero Structure on $\mathsf{Cats}^{\mathsf{cat}}_{-1}$

The Cartesian monoidal category $(\mathsf{Cats}^{\mathsf{cat}}_{-1}, \times, \mathsf{true})$ of Remark A.2.6 is more naturally thought of as a symmetric monoidal category with zero, and its monoidal product coincides on objects with the monoid with zero structure of the field with one element $\mathbb{F}_1$ of Monoids With Zero, Example 1.1.4.

Categories enriched in $(\mathsf{Cats}^{\mathsf{cat}}_{-1}, \times, \mathsf{true})$ are precisely the posets:

$$\mathsf{Pos} \cong \mathsf{Cats}_{\mathsf{Cats}^{\mathsf{cat}}_{-1}}.$$

A.3 0-Categories

Definition A.3.1 ▶ 0-Categories

A 0-**category** is a poset.[1]

[1] A 0-category is precisely a category enriched in the poset of (-1)-categories; see Remark A.2.7.

Definition A.3.2 ▶ 0-Groupoids

A 0-**groupoid** is a 0-category in which every morphism is invertible.[1]

[1] That is, a *set*.

A.4 Setoids

Definition A.4.1 ▶ Setoids

A **setoid** is a pair $(X, \sim)$

Proposition A.4.2 ▶ Properties of Setoids

1. *Equivalence With Sets.* We have an equivalence of categories $\mathsf{Setd} \cong \mathsf{Sets}$.

Proof A.4.3 ▶ Proof of Proposition A.4.2

Item 1: Equivalence With Sets

A.5 Table of Analogies Between Set Theory and Category Theory

Here we record some analogies between notions in set theory and category theory. Note that the analogies relating to presheaves relate equally well to copresheaves, as the opposite X^{op} of a set X is just X again.

Set Theory	Category Theory
Enrichment in {true, false}	Enrichment in Sets
Set X	Category $\mathcal{C}$
Element $x \in X$	Object $X \in \mathrm{Obj}(\mathcal{C})$
Function	Functor
Function $X \longrightarrow \{\mathsf{true}, \mathsf{false}\}$	Functor $\mathcal{C} \longrightarrow \mathsf{Sets}$
Function $X \longrightarrow \{\mathsf{true}, \mathsf{false}\}$	Presheaf $\mathcal{C}^{\mathsf{op}} \longrightarrow \mathsf{Sets}$
Characteristic function $\chi_{\{x\}}$	Representable presheaf h_X
Characteristic embedding $\chi_{(-)} \colon X \hookrightarrow \mathcal{P}(X)$	Yoneda embedding $よ \colon \mathcal{C}^{\mathsf{op}} \hookrightarrow \mathsf{PSh}(\mathcal{C})$
Characteristic relation $\chi_X(-_1, -_2)$	Hom profunctor $\mathrm{Hom}_{\mathcal{C}}(-_1, -_2)$
The Yoneda lemma for sets $\chi^{\mathsf{Pos}}_{\mathcal{P}(X)}(\chi_x, \chi_U) = \chi_U(x)$	The Yoneda lemma for categories $\mathrm{Nat}(h_X, \mathcal{F}) \cong \mathcal{F}(X)$
The characteristic embedding is fully faithful, $\chi^{\mathsf{Pos}}_{\mathcal{P}(X)}(\chi_x, \chi_y) = \chi_X(x, y)$	The Yoneda embedding is fully faithful, $\mathrm{Nat}(h_X, h_Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, Y)$
Powerset $\mathcal{P}(X)$	Presheaf category $\mathsf{PSh}(\mathcal{C})$
Relation $R \colon X \times Y \longrightarrow \{\mathsf{true}, \mathsf{false}\}$	Profunctor $\mathfrak{p} \colon \mathcal{D}^{\mathsf{op}} \times \mathcal{C} \longrightarrow \mathsf{Sets}$
Relation $R \colon X \longrightarrow \mathcal{P}(Y)$	Profunctor $\mathfrak{p} \colon \mathcal{C} \longrightarrow \mathsf{PSh}(\mathcal{D})$
Relation as a cocontinuous morphism of posets $R \colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(Y), \subset)$	Profunctor as a colimit-preserving functor $\mathfrak{p} \colon \mathsf{PSh}(\mathcal{C}) \to \mathsf{PSh}(\mathcal{D})$
Restricted comprehension	Categories of elements
Direct image function $f_* \colon \mathcal{P}(X) \to \mathcal{P}(Y)$	Inverse image functor $f^{-1} \colon \mathsf{PSh}(\mathcal{C}) \to \mathsf{PSh}(\mathcal{D})$
Inverse image function $f^{-1} \colon \mathcal{P}(Y) \to \mathcal{P}(X)$	Direct image functor $f_* \colon \mathsf{PSh}(\mathcal{D}) \to \mathsf{PSh}(\mathcal{C})$
Direct image with compact support function $f_! \colon \mathcal{P}(X) \to \mathcal{P}(Y)$	Direct image with compact support functor $f_! \colon \mathsf{PSh}(\mathcal{C}) \to \mathsf{PSh}(\mathcal{D})$

B Miscellany

B.1 Grothendieck Universes

B.1.1 Foundations

Definition B.1.1 ▶ Grothendieck Universes

A **Grothendieck universe**[1] is a nonempty set $\mathcal{U}$ satisfying the following conditions:

1. If $x \in \mathcal{U}$ and $y \in x$, then $y \in \mathcal{U}$.
2. If $x, y \in \mathcal{U}$, then $\{x, y\} \in \mathcal{U}$.
3. If $x \in \mathcal{U}$, then $\mathcal{P}(x) \in \mathcal{U}$.
4. If $I \in \mathcal{U}$ and, for each $\alpha \in I$, we have $x_\alpha \in \mathcal{U}$, then $\bigcup_{\alpha \in I} X_\alpha \in \mathcal{U}$.

[1] Or simply a **universe**.

Example B.1.2 ▶ The Universe Containing the Empty Set

The universe generated by the empty set is the universe $\mathcal{U}_\emptyset$ defined by

$$\mathcal{U}_\emptyset \overset{\text{def}}{=} \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \ldots\}.$$

Proposition B.1.3 ▶ Universes Are Closed Under Set-Theoretical Constructions

Grothendieck universes are closed under unions, formation of ordered pairs, Cartesian products, and exponentiation[1]

[1] I.e. taking Hom's.

Proof B.1.4 ▶ Proof of Proposition B.1.3

Clear. ∎

Axiom B.1.5 ▶ Existence of Grothendieck Universes

Every set X is contained in some universe $\mathcal{U}_X$.

Definition B.1.6 ▶ Tarski–Grothendieck Set Theory

Proposition B.1.7 ▶ Grothendieck Universes and Inaccessible Cardinals

The following statements are equivalent:

1. Grothendieck Universes exist (Axiom B.1.5).
2. Inaccessible Cardinals exist.

Proof B.1.8 ▶

Omitted.

B.1.2 $\mathcal{U}$-Small Sets

Let $\mathcal{U}$ be a Grothendieck universe.

Definition B.1.9 ▶ $\mathcal{U}$-Small Sets

A set is **$\mathcal{U}$-small** if it is isomorphic to an element of $\mathcal{U}$.

B.2 Foundations

Definition B.2.1 ▶ The ETCS Axioms

The **ETCS axioms** state that there exists a category Sets satisfying the following conditions:

1. *Finite Bicompleteness.* The category Sets is finitely bicomplete, admitting:
   (a) *Initial Objects.* An initial object $\emptyset$;
   (b) *Terminal Objects.* A terminal object pt;
   (c) *Binary Products.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, a product $A \times B$;
   (d) *Binary Coproducts.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, a coproduct $A \coprod B$;
   (e) *Equalisers.* For each parallel pair $f, g \colon A \rightrightarrows B$ of morphisms of Sets, an equaliser $\mathrm{Eq}(f, g)$;
   (f) *Coequalisers.* For each parallel pair $f, g \colon A \rightrightarrows B$ of morphisms of Sets, a coequaliser $\mathrm{CoEq}(f, g)$;

2. *Exponentials.* The category Sets is Cartesian closed, add

C Other Chapters

References

[BB10] John C. Baez and Michael Shulman. “Lectures on n-Categories and Cohomology”. In: *Towards higher categories*. Vol. 152. IMA Vol. Math. Appl. Springer, New York, 2010, pp. 1–68. DOI: 10.1007/978-1-4419-1524-5_1. URL: https://doi.org/10.1007/978-1-4419-1524-5_1 (cit. on pp. 9, 10).

[nLab23] The nLab Authors. *Negative Thinking*. 2023. URL: https://ncatlab.org/nlab/show/negative+thinking (cit. on p. 1).