Posets

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Introduction

This chapter contains some material about posets and constructions with them. Notably, it contains:

- A basic discussion of posets, constructions with them, and co/limits inside posets (Sections 1 to 4)
- A discussion of so-called *relative preorders* from a set X to a set Y. These are supposed to be an extension of the notion of a preorder $\preceq_X : X \nrightarrow X$ on a set X but where we allow the source and target of $\preceq_X$ to be entirely different sets.

  The basic idea is that we may view preorders as precisely the monads in Rel, so *relative preorders* are to be defined as *relative monads* in Rel in the sense of [nLab23].

  Thus, if you're interested in relative monads, you might like reading Section A.

Contents

1 Preorders and Partial Orders

1.1 Preorders

Let A be a set.

Definition 1.1.1 ▶ Preorders

A **preorder on** A is equivalently:[1]

- An $\mathbb{E}_1$-monoid in $(\mathrm{N}_\bullet(\mathbf{Rel}(A, A)), \chi_A)$;
- A monoid in $(\mathbf{Rel}(A, A), \chi_A)$.

[1] Note that since **Rel**(A, A) is posetal, being a preorder is a property of a relation, instead of a structure.

Remark 1.1.2 ▶ Unwinding Definition 1.1.1

In detail, a relation R on A is a **preorder** if there exists

- *The Multiplication Inclusion.* An inclusion

  $$\mu_R \colon R \diamond R \subset R$$

  of relations in $\mathbf{Rel}(A, A)$, i.e. if, for each $a, c \in A$, we have:

  $(\star)$ If $a \sim_R b$ and $b \sim_R c$, then $a \sim_R c$.

- *The Unit Inclusion.* An inclusion

  $$\eta_R \colon \chi_A \subset R$$

  of relations in $\mathbf{Rel}(A, A)$, i.e. if, for each $a \in A$, we have $a \sim_R a$.

Definition 1.1.3 ▶ The Po/Set of Preorders on a Set

Let A be a set.

1. The **set of preorders on** A is the subset $\mathrm{POrd}(A, A)$ of $\mathrm{Rel}(A, A)$ spanned by the preorders.
2. The **poset of preorders on** A is is the subposet $\mathbf{POrd}(A, A)$ of $\mathbf{Rel}(A, A)$ spanned by the preorders.

1.2 The Preorder Associated to a Relation

Let R be a relation on A.

Definition 1.2.1 ▶ The Preorder Associated to a Relation

The **preorder associated to** R is the preorder $\sim_R^{\mathrm{pord}}$[1] satisfying the following universal property:[2]

(UP) Given another preorder $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{pord}} \subset \sim_S$.

[1] *Further Notation:* Also written R^{pord}.

[2] *Slogan:* The preorder associated to R is the smallest preorder containing R.

Construction 1.2.2 ▶ The Preorder Associated to a Relation

Concretely, $\sim_R^{\mathrm{pord}}$ is the free monoid on R in $(\mathbf{Rel}(A, A), \diamond, \chi_A)$[1], being given by

$$\begin{aligned} R^{\mathrm{pord}} &\overset{\mathrm{def}}{=} \coprod_{n=0}^{\infty} R^{\diamond n} \\ &\overset{\mathrm{def}}{=} \Delta_A \cup \bigcup_{n=1}^{\infty} R^{\diamond n} \\ &\overset{\mathrm{def}}{=} \left\{ (a, b) \in A \times B \;\middle|\; \begin{array}{l} \text{we have } a = b \text{ or there exist} \\ (x_1, \ldots, x_n) \in R^{\times n} \text{ such that} \\ a \sim_R x_1 \sim_R \cdots \sim_R x_n \sim_R b \end{array} \right\}. \end{aligned}$$

[1] Or, equivalently, the free $\mathbb{E}_1$-monoid on R in $(\mathsf{N}_\bullet(\mathbf{Rel}(A, A)), \diamond, \chi_A)$.

Proof 1.2.3 ▶ Proof of Construction 1.2.2

Clear. ∎

Proposition 1.2.4 ▶ Properties of the Preorder Associated to Relation

Let R be a relation on A.

1. *Adjointness.* We have an adjunction

$$\left((-)^{\mathrm{pord}} \dashv 忘\right)\colon \quad \mathbf{Rel}(A, A) \underset{忘}{\overset{(-)^{\mathrm{pord}}}{\rightleftarrows}} \mathbf{POrd}(A, A),$$

witnessed by a bijection of sets

$$\mathbf{POrd}\left(\sim_R^{\mathrm{pord}}, \sim_S\right) \cong \mathbf{Rel}(\sim_R, \sim_S),$$

natural in $\sim_R \in \mathrm{Obj}(\mathbf{POrd}(A, A))$ and $\sim_S \in \mathrm{Obj}(\mathbf{Rel}(A, A))$.

2. *The Associated Preorder of a Preorder.* If R is partial order, then $R^{\mathrm{pord}} = R$.

3. *Idempotency.* We have

$$\left(R^{\mathrm{pord}}\right)^{\mathrm{pord}} = R^{\mathrm{pord}}.$$

Proof 1.2.5 ▶ Proof of Proposition 1.2.4

Item 1: Adjointness

This is a rephrasing of the universal property of the preorder associated to a relation, stated in Definition 1.2.1.

Item 2: The Associated Preorder of a Preorder

Clear.

Item 3: Idempotency

Clear. ∎

1.3 Partial Orders

Let X be a set.

Definition 1.3.1 ▶ Partial Orders

A **partial order on** X is a preorder $\preceq_X$ on X satisfying the following condition:

(★) For each $x, y \in X$, if $x \preceq_X y$ and $y \preceq_X x$, then $x = y$.

Remark 1.3.2 ▶ Unwinding Definition 1.3.1

In detail, a **partial order** is a relation $\preceq_X : X \nrightarrow X$ on X satisfying the following conditions:

1. *Reflexivity.* For each $x \in X$, we have $x \preceq_X x$.
2. *Transitivity.* For each $x, y, z \in X$, if $x \preceq_X y$ and $y \preceq_X z$, then $x \preceq_X z$.
3. *Antisymmetry.* For each $x, y \in X$, if $x \preceq_X y$ and $y \preceq_X x$, then $x = y$.

Definition 1.3.3 ▶ The Po/Set of Partial Orders on a Set

Let X be a set.

1. The **set of partial orders relations on** X is the subset $\mathrm{PartOrd}(X, X)$ of $\mathrm{Rel}(X, X)$ spanned by the partial orders.
2. The **poset of partial orders relations on** X is is the subposet $\mathbf{PartOrd}(X, X)$ of $\mathbf{Rel}(X, X)$ spanned by the partial orders.

1.4 The Partial Orders Associated to a Relation

Let R be a relation on X.

Definition 1.4.1 ▶ The Partial Order Associated to a Relation

The **partial order associated to** R is the partial order $\sim_R^{\mathrm{ptord}}$[1] satisfying the following universal property:[2]

(UP) Given another partial order $\sim_S$ on X such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{ptord}} \subset \sim_S$.

[1] *Further Notation:* Also written R^{ptord}.
[2] *Slogan:* The partial order associated to R is the smallest partial order containing R.

Construction 1.4.2 ▶ The Partial Order Associated to a Relation

Concretely, $\sim_R^{\mathrm{ptord}}$ is the partial order on X defined by

$$\begin{aligned} R^{\mathrm{ptord}} &\overset{\mathrm{def}}{=} \left(R^{\mathrm{antisymm}}\right)^{\mathrm{pord}} \\ &\cong (R/\sim)^{\mathrm{pord}} \\ &\overset{\mathrm{def}}{=} \Delta_A \cup \bigcup_{n=1}^{\infty} (R/\sim)^{\diamond n}, \end{aligned}$$

where $\sim$ is the equivalence relation on R obtained by declaring $a \sim b$ iff $a \sim_R b$ and $b \sim_R a$.

Proof 1.4.3 ▶ Proof of Construction 1.4.2

Clear. ∎

Proposition 1.4.4 ▶ Properties of the Partial Order Associated to Relation

Let R be a relation on X.

1. *Adjointness.* We have an adjunction

$$\left((-)^{\mathrm{ptord}} \dashv 忘\right)\colon \quad \mathbf{Rel}(X, X) \underset{忘}{\overset{(-)^{\mathrm{ptord}}}{\rightleftarrows}} \mathbf{PartOrd}(X, X),$$

witnessed by a bijection of sets

$$\mathbf{PartOrd}\left(\sim_R^{\mathrm{ptord}}, \sim_S\right) \cong \mathbf{Rel}(\sim_R, \sim_S),$$

natural in $\sim_R \in \mathrm{Obj}(\mathbf{PartOrd}(X, X))$ and $\sim_S \in \mathrm{Obj}(\mathbf{Rel}(X, X))$.

2. *The Associated Preorder of a Preorder.* If R is partial order, then $R^{\mathrm{ptord}} = R$.

3. *Idempotency.* We have

$$\left(R^{\mathrm{ptord}}\right)^{\mathrm{ptord}} = R^{\mathrm{ptord}}.$$

Proof 1.4.5 ▶ Proof of Proposition 1.4.4

Item 1: Adjointness

This is a rephrasing of the universal property of the partial order associated to a relation, stated in Definition 1.4.1.

Item 2: The Associated Preorder of a Preorder

Clear.

Item 3: Idempotency

Clear.

1.5 Total Orders

Let X be a set.

Definition 1.5.1 ▶ Total Orders

A **total order on** X is a partial order $\preceq_X$ on X satisfying the following condition:

($\star$) For each $x, y \in X$, we have either $x \preceq_X y$ or $y \preceq_X x$.

2 Posets

2.1 Foundations

Definition 2.1.1 ▶ Posets

A **poset** $(X, \preceq_X)$ consists of

- *The Underlying Set.* A set X, called the **underlying set of** $(X, \preceq_X)$;
- *The Partial Order.* A partial order

$$\preceq_X : X \times X \to \{\text{true}, \text{false}\}$$

on X, called the **partial order of** $(X, \preceq_X)$.

Example 2.1.2 ▶ Powersets

Given a set X, the pair $(\mathcal{P}(X), \subset)$ is a poset, as is $(\mathcal{P}(X), \supset)$.

Definition 2.1.3 ▶ The Posetal Characteristic Relation of a Poset

The **posetal characteristic relation** of a poset $(X, \preceq)$ is the relation

$$\chi^{\text{Pos}}_{(X,\preceq)} : X \times X \to \{\text{true}, \text{false}\}$$

on X defined by[1]

$$\chi^{\text{Pos}}_{(X,\preceq)}(x, y) \overset{\text{def}}{=} \begin{cases} \text{true} & \text{if } x \preceq y, \\ \text{false} & \text{if } x \npreceq y \end{cases}$$

for each $x, y \in X$.

[1] In other words, $\chi^{\text{Pos}}_{(X,\preceq)}$ is just the Hom of the posetal category associated to $(X, \preceq)$, defined by

$$\text{Hom}_{(X,\preceq)}(x, y) \overset{\text{def}}{=} \begin{cases} \text{pt} & \text{if } x \preceq y, \\ \emptyset & \text{if } x \npreceq y, \end{cases}$$

but one level lower in enrichment, as $\chi^{\text{Pos}}_{(X,\preceq)}$ takes values in $\{\text{true}, \text{false}\}$, instead of in $\{\text{pt}, \emptyset\} \subset$ Sets.

2.2 Morphisms of Posets

Let $(X, \preceq_X)$ and $(Y, \preceq_Y)$ be posets.

Definition 2.2.1 ▶ Morphisms of Posets

A **morphism of posets from** $(X, \preceq_X)$ **to** $(Y, \preceq_Y)$[1] is a function $f\colon X \to Y$ satisfying the following condition:

($\star$) *Monotonicity*. For each $x, y \in X$, if $x \preceq_X y$, then $f(x) \preceq_Y f(y)$.

[1] *Further Terminology*: Also called a **monotone function**.

2.3 Ideals of Posets

Let $(X, \preceq_X)$ be a poset.

Definition 2.3.1 ▶ Ideals of Posets

An **ideal of** $(X, \preceq_X)$ is a subset I of X satisfying the following conditions:

1. *Non-Emptiness*. We have $I \neq \emptyset$.
2. *Upward-Directedness*. For each $x, y \in I$, there exists some $c_{x,y} \in I$ such that:
   - We have $x \preceq_X c_{x,y}$.
   - We have $y \preceq_X c_{x,y}$.
3. *Downward-Closedness*. For each $x, y \in I$, if:
   - We have $y \in I$;
   - We have $x \preceq_X y$;

   then $x \in I$.

Remark 2.3.2 ▶ Alternative Axioms for Ideals of Lattices

If $(X, \preceq_X)$ is a lattice, then $I \subset X$ is an ideal of $(X, \preceq_X)$ iff the following conditions are satisfied:

1. *Containment of the Bottom Element*. We have $\perp \in I$.
2. *Closure Under Binary Joins*. If $x, y \in I$, then $x \vee y \in I$.
3. *Closure Under Binary Joins With Elements of X*. If $a \in X$ and $x \in I$, then $a \vee x \in I$.

Definition 2.3.3 ▶ Proper Ideals

An ideal I of $(X, \preceq_X)$ is **proper** if $I \neq X$.

Definition 2.3.4 ▶ Prime Ideals

An ideal I of $(X, \preceq_X)$ is **prime** if the following conditions are satisfied:

1. *Properness.* The ideal I is proper.
2. *Primality.* For each $x, y \in I$, if $x \vee y \in I$, then $x \in I$ or $y \in I$.

Definition 2.3.5 ▶ Completely Prime Ideals

An ideal I of a lattice $(X, \preceq_X)$ is **completely prime** if the following conditions are satisfied:

1. *Properness.* The ideal I is proper.
2. *Infinitary Primality.* For each $\{x_i\}_{i\in J} \in \mathcal{P}(I)$, if $\bigvee_{i\in J} x_i \in I$, then there exists some $i \in I$ such that $x_i \in I$.

Definition 2.3.6 ▶ Maximal Ideals

An ideal I of $(X, \preceq_X)$ is **maximal** if the following conditions are satisfied:

1. *Properness.* The ideal I is proper.
2. *Maximality.* Given another ideal J of X, if $I \subset J$, then $J = X$.

2.4 Filters on Posets

2.4.1 Foundations

Let $(X, \preceq_X)$ be a poset.

Definition 2.4.1 ▶ Filters on Posets

A **filter on** $(X, \preceq_X)$ is a subset F of X satisfying the following conditions:

1. *Non-Emptiness.* We have $F \neq \emptyset$.
2. *Upward-Closedness.* For each $x, y \in X$, if:

- We have $x \in F$;
- We have $x \preceq_X y$;

then $y \in F$.

3. *Downward-Directedness*. For each $x, y \in F$, there exists some $c_{x,y} \in F$ such that:
   - We have $c_{x,y} \preceq_X x$;
   - We have $c_{x,y} \preceq_X y$.

Remark 2.4.2 ▶ Alternative Axioms for Filters on Lattices

If $(X, \preceq_X)$ is a lattice, then $F \subset X$ is a filter on $(X, \preceq_X)$ iff the following conditions are satisfied:[1]

1. *Containment of the Top Element*. We have $\top \in F$.
2. *Closure Under Binary Meets*. If $x, y \in F$, then $x \wedge y \in F$.
3. *Closure Under Binary Joins With Elements of X*. If $a \in X$ and $x \in F$, then $a \vee x \in F$.

[1] These conditions are equivalent to the statement that $\chi_F \colon X \to \{\text{true}, \text{false}\}$ is a morphism of meet-semilattices.

2.4.2 Proper Filters

Let $(X, \preceq_X)$ be a poset.

Definition 2.4.3 ▶ Proper Filters

A filter F on X is **proper** if $F \neq X$.[1]

[1] *Further Terminology:* The filter X on X is called the **improper filter**.

2.4.3 Prime Filters

Let $(X, \preceq_X)$ be a lattice.

Definition 2.4.4 ▶ Prime Filters

A filter F on X is **prime** if $X \setminus F$ is an ideal of X.

Remark 2.4.5 ▶ Unwinding Definition 2.4.4

That is, F is **prime** if the following conditions are satisfied:

- *Properness.* We have $\bot \notin F$.
- *Primality.* For each $x, y \in X$, if $x \vee y \in F$, then $x \in F$ or $y \in F$.

2.4.4 Completely Prime Filters

Let $(X, \preceq_X)$ be a lattice.

Definition 2.4.6 ▶ Completely Prime Filters

A filter F on X is **completely prime** if the following conditions are satisfied:

- *Properness.* We have $\bot \notin F$.
- *Primality.* For each $\{x_i\}_{i \in I} \in \mathcal{PC}(X)$, if $\bigvee_{i \in I} x_i \in F$, then there exists some $i \in I$ such that $x_i \in F$.

2.4.5 Ultrafilters

3 Constructions With Posets

3.1 The Dual of a Poset

Let $(X, \preceq_X)$ be a poset.

Definition 3.1.1 ▶ The Dual of a Poset

The **dual of** $(X, \preceq_X)$ is the poset $(X^{\mathrm{op}}, \preceq_{X^{\mathrm{op}}})$ consisting of

- *The Underlying Set.* The set X^{op} defined by

$$X^{\mathrm{op}} \overset{\mathrm{def}}{=} X;$$

- *The Partial Order.* The partial order

$$\preceq_{X^{\mathrm{op}}} : X^{\mathrm{op}} \times X^{\mathrm{op}} \to \{\mathrm{true}, \mathrm{false}\}$$

on X^{op} defined by

$$x \preceq_{X^{\mathsf{op}}} y \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } y \preceq_X x, \\ \text{false} & \text{otherwise.} \end{cases}$$

Example 3.1.2 ▶ Dual of Powersets

Let X be a set. The dual of $(\mathcal{P}(X), \subset)$ is $(\mathcal{P}(X), \supset)$.

3.2 Products of Posets

Let $(X, \preceq_X)$ and $(Y, \preceq_Y)$ be posets.

Definition 3.2.1 ▶ Products of Posets

The **product of** $(X, \preceq_X)$ **and** $(Y, \preceq_Y)$ is the poset $(X \times Y, \preceq_{X\times Y})$ consisting of

- *The Underlying Set.* The Cartesian product $X \times Y$ of X and Y;
- *The Partial Order.* The partial order

$$\preceq_{X\times Y} \colon (X \times Y) \times (X \times Y) \to \{\text{true}, \text{false}\}$$

on $X \times Y$ defined by

$$(a, b) \preceq_{X\times Y} (x, y) \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } a \preceq_X x \text{ and } b \preceq_Y y, \\ \text{false} & \text{otherwise.} \end{cases}$$

3.3 Coproducts of Posets

Let $(X, \preceq_X)$ and $(Y, \preceq_Y)$ be posets.

Definition 3.3.1 ▶ Coproducts of Posets

The **coproduct of** $(X, \preceq_X)$ **and** $(Y, \preceq_Y)$ is the poset $\left(X \coprod Y, \preceq_{X \coprod Y}\right)$ consisting of

- *The Underlying Set.* The disjoint union $X \coprod Y$ of X and Y;

- *The Partial Order.* The partial order

$$\preceq_{X \coprod Y} : (X \coprod Y) \times (X \coprod Y) \to \{\text{true}, \text{false}\}$$

on $X \coprod Y$ defined by

$$x \preceq_{X \coprod Y} y \overset{\text{def}}{=} \begin{cases} \text{true} & \text{if } x, y \in X \text{ and } x \preceq_X y, \\ \text{true} & \text{if } x, y \in Y \text{ and } x \preceq_Y y, \\ \text{false} & \text{otherwise.} \end{cases}$$

3.4 The Tensor Product of Posets

3.4.1 Bilinear Morphisms of Posets

Let $(X, \preceq_X)$, $(Y, \preceq_Y)$, and $(Z, \preceq_Z)$ be posets.

Definition 3.4.1 ▶ Bilinear Morphisms of Posets

A **bilinear morphism of posets from** $(X \times Y, \preceq_{X \times Y})$ **to** $(Z, \preceq_Z)$ is a function $f : X \times Y \to Z$ satisfying the following conditions:

- For each $x, y \in X$ and each $z \in Y$, if $x \preceq_X y$, then $f(x, z) \preceq_Z f(y, z)$.
- For each $x \in X$ and each $y, z \in Y$, if $y \preceq_Y z$, then $f(x, y) \preceq_Z f(x, z)$.

3.4.2 The Tensor Product of Posets

Let $(X, \preceq_X)$ and $(Y, \preceq_Y)$ be posets.

Definition 3.4.2 ▶ The Tensor Product of Posets

The **tensor product of** $(X, \preceq_X)$ **and** $(Y, \preceq_Y)$ is the pair $(X \boxtimes Y, \iota)$ consisting of

- The poset $X \boxtimes Y$;
- The bilinear morphism of posets $\iota : X \times Y \to X \boxtimes Y$;

satisfying the following universal property:

(UP) Given another pair (Z, i) consisting of

- A poset Z;

- A bilinear morphism of posets $i\colon X \times Y \to Z$;

there exists a unique morphism of posets $X \boxtimes Y \xrightarrow{\exists!} Z$ making the diagram

$$\begin{array}{ccc} X \times Y & \xrightarrow{\iota} & X \boxtimes Y \\ & {\scriptstyle i}\searrow & \Big\downarrow {\scriptstyle \exists!} \\ & & Z \end{array}$$

commute.

Construction 3.4.3 ▶ Tensor Products of Posets

Concretely, the **tensor product of** X **and** Y is the poset $X \boxtimes Y$ defined by

$$X \boxtimes Y \overset{\text{def}}{=} X_{\text{cat}} \times Y_{\text{disc}} \coprod_{X_{\text{disc}} \times Y_{\text{disc}}} X_{\text{disc}} \times Y_{\text{cat}}, \qquad \begin{array}{ccc} X \boxtimes Y & \longleftarrow & X_{\text{cat}} \times Y_{\text{disc}} \\ \Big\uparrow & \ulcorner & \Big\uparrow \\ X_{\text{disc}} \times Y_{\text{cat}} & \longleftarrow & X_{\text{disc}} \times Y_{\text{disc}}. \end{array}$$

3.5 Internal Homs

Let $(X, \preceq_X)$ and $(Y, \preceq_Y)$ be posets.

Definition 3.5.1 ▶ Internal Homs of Posets

The **internal Hom of posets from** $(X, \preceq_X)$ **to** $(Y, \preceq_Y)$ is the poset $\mathbf{Pos}((X, \preceq_X), (Y, \preceq_Y))$[1] consisting of

- *The Underlying Set.* The set $\mathrm{Pos}((X, \preceq_X), (Y, \preceq_Y))$ of morphisms of posets from $(X, \preceq_X)$ to $(Y, \preceq_Y)$;
- *The Partial Order.* The partial order

  $$\preceq_{\mathbf{Pos}(X,Y)}\colon \mathrm{Pos}(X, Y) \times \mathrm{Pos}(X, Y) \to \{\text{true}, \text{false}\}$$

  on $\mathrm{Pos}(X, Y)$ defined by[2]

  $$f \preceq g \overset{\text{def}}{=} \begin{cases} \text{true} & \text{if } f(x) \preceq g(x) \text{ for each } x \in X, \\ \text{false} & \text{otherwise} \end{cases}$$

for each $f, g \in \mathsf{Pos}(f, g)$.

[1] *Further Notation:* Also written simply $\mathbf{Pos}(X, Y)$.

[2] *Further Terminology:* Due to its definition, $\preceq_{\mathbf{Pos}(X,Y)}$ is called the **pointwise partial order** on $\mathsf{Pos}(X, Y)$.

4 Co/Limits in Posets

4.1 Initial Elements

Let $(X, \preceq_X)$ be a poset.

Definition 4.1.1 ▶ The Initial Element of a Poset

The **initial element of** $(X, \preceq_X)$[1] is, if it exists, the element $\perp$ of X satisfying the following condition:

($\star$) For each $x \in X$, we have $\perp \preceq_X x$.

[1] *Further Terminology:* Also called the **bottom element of** $(X, \preceq_X)$.

Example 4.1.2 ▶ The Initial Element of a Powerset

Let X be a set. The initial element of $(\mathcal{P}(X), \subset)$ is given by $\emptyset$.

4.2 Final Elements

Let $(X, \preceq_X)$ be a poset.

Definition 4.2.1 ▶ The Final Element of a Poset

The **final element of** $(X, \preceq_X)$[1] is, if it exists, the element $\top$ of X satisfying the following condition:

($\star$) For each $x \in X$, we have $x \preceq_X \top$.

[1] *Further Terminology:* Also called the **top element of** $(X, \preceq_X)$.

Example 4.2.2 ▶ The Final Element of a Powerset

Let X be a set. The final element of $(\mathcal{P}(X), \subset)$ is given by X.

4.3 Binary Joins

Let $(X, \preceq_X)$ be a poset and let $x, y \in X$.

Definition 4.3.1 ▶ Binary Joins in a Poset

The **binary join of x and y in** $(X, \preceq_X)$ is, if it exists, the element $x \vee y$ of X satisfying the following conditions:

1. We have $x \preceq_X x \vee y$ and $y \preceq_X x \vee y$.
2. For each $s \in X$, if $x \preceq_X s$ and $y \preceq_X s$, then $x \vee y \preceq_X s$.

Example 4.3.2 ▶ Binary Joins in Powersets

Let X be a set. The binary join of U and V in $(\mathcal{P}(X), \subset)$ is given by $U \cup V$.

4.4 Joins of Families

Let $(X, \preceq_X)$ be a poset and let $\{x_i\}_{i \in I}$ be a family of elements of X.

Definition 4.4.1 ▶ Joins of Families of Elements in a Poset

The **join of** $\{x_i\}_{i \in I}$ **in** $(X, \preceq_X)$ is, if it exists, the element $\bigvee_{i \in I} x_i$ of X satisfying the following conditions:

1. For each $i \in I$, we have $x_i \preceq_X \bigvee_{i \in I} x_i$.
2. For each $s \in X$, the following condition is satisfied:

   (★) If, for each $i \in I$, we have $x_i \preceq_X s$, then $\bigvee_{i \in I} x_i \preceq_X s$.

Example 4.4.2 ▶ Joins of Empty Families

The meet $\bigvee_{i \in \emptyset} x_i$ of the empty family is given by (if it exists) the bottom element $\perp$ of $(X, \preceq_X)$.

Example 4.4.3 ▶ Joins of Families in Powersets

Let X be a set. The join of a family $\{U_i\}_{i \in I}$ in $(\mathcal{P}(X), \subset)$ is given by $\bigcup_{i \in I} U_i$.

4.5 Binary Meets

Let $(X, \preceq_X)$ be a poset and let $x, y \in X$.

Definition 4.5.1 ▶ Binary Meets in a Poset

The **binary meet of x and y in $(X, \preceq_X)$** is, if it exists, the element $x \wedge y$ of X satisfying the following conditions:

1. We have $x \wedge y \preceq_X x$ and $x \wedge y \preceq_X y$.
2. For each $a \in X$, if $a \preceq_X x$ and $a \preceq_X y$, then $a \preceq_X x \wedge y$.

Example 4.5.2 ▶ Binary Meets in Powersets

Let X be a set. The binary meet of U and V in $(\mathcal{P}(X), \subset)$ is given by $U \cap V$.

4.6 Meets of Families

Let $(X, \preceq_X)$ be a poset and let $\{x_i\}_{i \in I}$ be a family of elements of X.

Definition 4.6.1 ▶ Meets of Families of Elements in a Poset

The **meet of $\{x_i\}_{i \in I}$ in $(X, \preceq_X)$** is, if it exists, the element $\bigwedge_{i \in I} x_i$ of X satisfying the following conditions:

1. For each $i \in I$, we have $\bigwedge_{i \in I} x_i \preceq_X x_i$.
2. For each $s \in X$, the following condition is satisfied:

   (★) If, for each $i \in I$, we have $s \preceq_X x_i$, then $s \preceq_X \bigwedge_{i \in I} x_i$.

Example 4.6.2 ▶ Meets of Empty Families

The meet $\bigwedge_{i \in \emptyset} x_i$ of the empty family is given by (if it exists) the top element $\top$ of $(X, \preceq_X)$.

Example 4.6.3 ▶ Meets of Families in Powersets

Let X be a set. The meet of a family $\{U_i\}_{i \in I}$ in $(\mathcal{P}(X), \subset)$ is given by $\bigcap_{i \in I} U_i$.

4.7 Lattices

Definition 4.7.1 ▶ Lattices

Let $(X, \preceq_X)$ be a poset.

1. The poset $(X, \preceq_X)$ is a **join-semilattice** if it has a bottom element and binary joins.[1]
2. The poset $(X, \preceq_X)$ is a **meet-semilattice** if it has a top element and binary meets.[2]
3. The poset $(X, \preceq_X)$ is a **suplattice** if it has joins of arbitrary families.
4. The poset $(X, \preceq_X)$ is an **inflattice** if it has meets of arbitrary families.
5. The poset $(X, \preceq_X)$ is a **lattice** if it is both a join-semilattice and a meet-semilattice.
6. The poset $(X, \preceq_X)$ is a **complete lattice** if it is both a lattice and an inflattice.
7. The poset $(X, \preceq_X)$ is a **cocomplete lattice** if it is both a lattice and a suplattice.
8. The poset $(X, \preceq_X)$ is a **bicomplete lattice** if it is both a complete lattice and a cocomplete lattice.

[1] This is equivalent to having joins of finite families.
[2] This is equivalent to having meets of finite families.

Appendices

A Relative Preorders

A.1 The Left Skew Monoidal Structure on Rel(A, B)

Let A and B be sets and let $J\colon A \nrightarrow B$ be a relation.

Definition A.1.1 ▶ The Left J-Skew Monoidal Structure on $\mathbf{Rel}(A, B)$

The **left J-skew monoidal category of functors from A to B** is the left skew monoidal category $\big(\mathbf{Rel}(A, B), \lhd_J, J\big)$ consisting of

- *The Underlying Category.* The category $\mathbf{Rel}(A, B)$ of relations from A to B;
- *The Skew Monoidal Product.* The functor

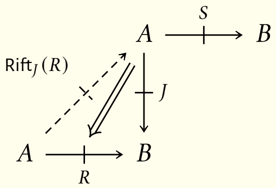
$$\lhd_J\colon \ \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \longrightarrow \mathbf{Rel}(A, B)$$

  from $\mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B)$ to $\mathbf{Rel}(A, B)$, called the **left J-skew monoidal product of relations from A to B**, where

  - *Action on Objects.* For each $R, S \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, we have

$$S \lhd_J R \overset{\mathrm{def}}{=} S \diamond \mathrm{Rift}_J(R),$$

    where $S \diamond \mathrm{Rift}_J(R)$ is the composition



    in $\mathbf{Rel}$;
  - *Action on Morphisms.* For each $R, S, R', S' \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, the action on Hom-sets

$$(\lhd_J)_{(G,F),(G',F')}\colon \ \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S, S') \times \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, R') \to \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S \lhd_J R, S' \lhd_J R')$$

    of $\lhd_J$ at $((R, S), (R', S'))$ is defined by

$$\beta \lhd_J \alpha \overset{\mathrm{def}}{=} \beta \diamond \mathrm{Rift}_J(\alpha),$$

    for each $\beta \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S, S')$ and each $\alpha \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, R')$;

- *The Skew Monoidal Unit.* The functor

$$\mathbb{1}^{\mathbf{Rel}(A,B)}\colon \ \mathrm{pt} \longrightarrow \mathbf{Rel}(A, B)$$

  defined by

$$\mathbb{1}_{\mathbf{Rel}(A,B)} \overset{\mathrm{def}}{=} J;$$

- *The Skew Associators*. The natural transformation

$$\alpha^{\mathbf{Rel}(A,B)}\colon \lhd_J \circ (\lhd_J \times \mathrm{id}) \Longrightarrow \lhd_J \circ (\mathrm{id} \times \lhd_J),$$

whose component

$$\alpha^{\mathbf{Rel}(A,B)}_{T,S,R}\colon \underbrace{(T \lhd_J S) \lhd_J R}_{\overset{\mathrm{def}}{=} T \diamond \mathrm{Rift}_J(S) \diamond \mathrm{Rift}_J(R)} \subset \underbrace{T \lhd_J (S \lhd_J R)}_{\overset{\mathrm{def}}{=} T \diamond \mathrm{Rift}_J(S \diamond \mathrm{Rift}_J(R))}$$

at (T, S, R) is given by

$$\alpha^{\mathbf{Rel}(A,B)}_{T,S,R} \overset{\mathrm{def}}{=} \mathrm{id}_T \diamond \gamma,$$

where

$$\gamma\colon \mathrm{Rift}_J(S) \diamond \mathrm{Rift}_J(R) \subset \mathrm{Rift}_J(S \diamond \mathrm{Rift}_J(R))$$

is the inclusion adjunct to the inclusion

$$\underbrace{J \diamond \mathrm{Rift}_J(S) \diamond \mathrm{Rift}_J(R)}_{\overset{\mathrm{def}}{=} J_*(\mathrm{Rift}_J(S) \diamond \mathrm{Rift}_J(R))} \overset{\epsilon_S \star \mathrm{id}_{\mathrm{Rift}_J(R)}}{\subset} S \diamond \mathrm{Rift}_J(R)$$

under the adjunction $J_* \dashv \mathrm{Rift}_J$, where $\epsilon\colon J \diamond \mathrm{Rift}_J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J_* \dashv \mathrm{Rift}_J$;

- *The Skew Left Unitors*. The natural transformation

$$\lambda^{\mathbf{Rel}(A,B)}\colon \lhd_J \circ \left(\mathbb{1}^{\mathbf{Rel}(A,B)} \times \mathrm{id}\right) \Longrightarrow \mathrm{id},$$

whose component

$$\lambda^{\mathbf{Rel}(A,B)}_R\colon \underbrace{J \lhd_J R}_{\overset{\mathrm{def}}{=} J \diamond \mathrm{Rift}_J(R)} \subset R$$

at R is given by

$$\lambda^{\mathbf{Rel}(A,B)}_R \overset{\mathrm{def}}{=} \epsilon_R,$$

where $\epsilon\colon J \diamond \mathrm{Rift}_J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J_* \dashv \mathrm{Rift}_J$;

- *The Skew Right Unitors.* The natural transformation

$$\rho^{\mathbf{Rel}(A,B)}\colon \mathrm{id} \Longrightarrow \lhd_J \circ \left(\mathrm{id} \times \mathbb{1}^{\mathbf{Rel}(A,B)}\right),$$

whose component

$$\rho^{\mathbf{Rel}(A,B)}_R\colon R \subset \underbrace{R \lhd_J J}_{\overset{\text{def}}{=} R \diamond \mathrm{Rift}_J(J)}$$

at R is given by

$$\rho^{\mathbf{Rel}(A,B)}_R \overset{\text{def}}{=} \mathrm{id}_R \star \sigma,$$

where $\sigma\colon \mathrm{id}_A \Longrightarrow \mathrm{Rift}_J(J)$ is the universal transformation included in the data of the right Kan lift $\mathrm{Rift}_J(J)$.

A.2 Left Relative Preorders

Let A and B be sets and let $J\colon A \nrightarrow B$ be a relation.

Definition A.2.1 ▶ Left J-Relative Preorders

A **left J-relative preorder from A to B** is equivalently:

- An $\mathbb{E}_1$-skew monoid in $(\mathrm{N}_\bullet(\mathbf{Rel}(A, A)), \lhd_J, J)$;
- A skew monoid in $(\mathbf{Rel}(A, C), \lhd_J, J)$.

Remark A.2.2 ▶ Unwinding Definition A.2.1, I

In detail, a **left J-relative preorder** (R, μ_R, η_R) **from** A **to** B consists of

- *The Underlying Relation.* A relation

$$R\colon A \nrightarrow B,$$

called the **underlying relation of** (R, μ_R, η_R);

- *The Multiplication Inclusion.* An inclusion of relations

$$\mu_R\colon R \lhd_J R \subset R,$$

called the **multiplication of** (R, μ_R, η_R);

- *The Unit Inclusion*. An inclusion of relations

$$\eta_R : J \subset R,$$

called the **unit of** (R, μ_R, η_R).

Remark A.2.3 ▶ Unwinding Definition A.2.1, II

In other words, a **left J-relative preorder from A to B** is a relation $R\colon A \nrightarrow B$ from A to B satisfying the following conditions:

1. *J-Transitivity*. For each $a \in A$ and each $c \in B$, the following condition is satisfied:[1]

   (★) If there exists some $b \in A$ such that:
   - For each $x \in B$, if $b \sim_J x$, then $a \sim_R x$;[2]
   - We have $b \sim_R c$;

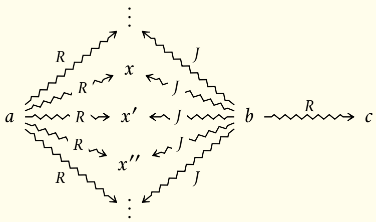
   then $a \sim_R c$.

2. *J-Unitality*. For each $a \in A$ and each $b \in B$, the following condition is satisfied:

   (★) If $a \sim_J b$, then $a \sim_R b$.

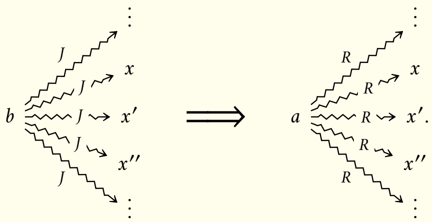
---

[1] If we have



then $a \sim_R c$.

[2] Illustration:



A.3 The Right Skew Monoidal Structure on Rel(A, B)

Let A and B be sets and let $J: A \nrightarrow B$ be a relation.

Definition A.3.1 ▶ The Right J-Skew Monoidal Structure on Rel(A, B)

The **right J-skew monoidal category of functors from A to B** is the right skew monoidal category consisting of

- *The Underlying Category.* The category **Rel**(A, B) of relations from A to B;
- *The Skew Monoidal Product.* The functor

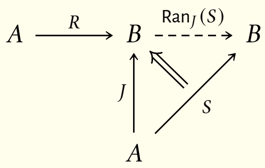
  $$\triangleright_J : \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \longrightarrow \mathbf{Rel}(A, B)$$

  from **Rel**$(A, B) \times$ **Rel**(A, B) to **Rel**(A, B), called the **right J-skew monoidal product of functors from A to B**, where

  - *Action on Objects.* For each $R, S \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, we have

    $$S \triangleright_J R \stackrel{\text{def}}{=} \mathrm{Ran}_J(S) \diamond R,$$

    where $\mathrm{Ran}_J(S) \diamond R$ is the composition

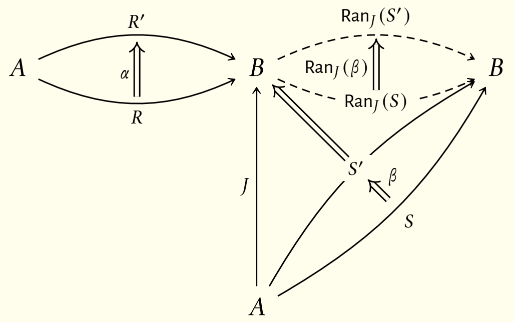
    in Cats;
  - *Action on Morphisms.* For each $R, S, R', S' \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, the action on Hom-sets

    $$(\triangleright_J)_{(S,R),(S',R')} : \mathrm{Nat}(S, S') \times \mathrm{Nat}(R, R') \to \mathrm{Nat}(S \triangleright_J R, S' \triangleright_J R')$$

    of $\triangleright_J$ at $((S, R), (S', R'))$ is defined by

    $$\beta \triangleright_J \alpha \stackrel{\text{def}}{=} \mathrm{Ran}_J(\beta) \diamond \alpha,$$

where $\mathrm{Ran}_J(\beta) \diamond \alpha$ is the horizontal composition



in Cats;

- *The Skew Monoidal Unit.* The functor

$$\Bbbk^{\mathbf{Rel}(A,B)} \colon \mathrm{pt} \longrightarrow \mathbf{Rel}(A, B)$$

defined by

$$\Bbbk_{\mathbf{Rel}(A,B)} \stackrel{\text{def}}{=} J;$$

- *The Skew Associators.* The natural transformation

$$\alpha^{\mathbf{Rel}(A,B)} \colon \triangleright_J \circ (\mathrm{id} \times \triangleright_J) \Longrightarrow \triangleright_J \circ (\triangleright_J \times \mathrm{id}),$$

whose component

$$\alpha^{\mathbf{Rel}(A,B)}_{T,S,R} \colon \underbrace{T \triangleright_J (S \triangleright_J R)}_{\stackrel{\text{def}}{=} \mathrm{Ran}_J(T) \diamond (\mathrm{Ran}_J(S) \diamond R)} \subset \underbrace{(T \triangleright_J S) \triangleright_J R}_{\stackrel{\text{def}}{=} \mathrm{Ran}_J(\mathrm{Ran}_J(T) \diamond S) \diamond R}$$

at (T, S, R) is given by

$$\alpha^{\mathbf{Rel}(A,B)}_{T,S,R} \stackrel{\text{def}}{=} \gamma \diamond \mathrm{id}_R,$$

where

$$\gamma \colon \mathrm{Ran}_J(T) \diamond \mathrm{Ran}_J(S) \subset \mathrm{Ran}_J(\mathrm{Ran}_J(T) \diamond S)$$

is the inclusion adjunct to the inclusion

$$\underbrace{\mathrm{Ran}_J(T) \diamond \mathrm{Ran}_J(S) \diamond J}_{\stackrel{\text{def}}{=} J^*(\mathrm{Ran}_J(T) \diamond \mathrm{Ran}_J(S))} \xRightarrow{\mathrm{id}_{\mathrm{Ran}_J(T)} \diamond \epsilon_S} \mathrm{Ran}_J(T) \diamond S$$

under the adjunction $J^* \dashv \mathsf{Ran}_J$, where $\epsilon\colon \mathsf{Ran}_J \diamond J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J^* \dashv \mathsf{Ran}_J$;

- *The Skew Left Unitors.* The natural transformation

$$\lambda^A\colon \mathrm{id} \Longrightarrow \triangleright_J \circ \left(\mathbb{1}^{\mathbf{Rel}(A,B)} \times \mathrm{id}\right),$$

whose component

$$\lambda^{\mathbf{Rel}(A,B)}_R\colon R \subset \underbrace{J \triangleright_J R}_{\stackrel{\text{def}}{=}\mathsf{Ran}_J(J)\diamond R}$$

at R is given by

$$\lambda^{\mathbf{Rel}(A,B)}_R \stackrel{\text{def}}{=} \sigma \diamond \mathrm{id}_R,$$

where $\sigma\colon \mathrm{id}_B \Longrightarrow \mathsf{Ran}_J(J)$ is the unit of the codensity monad of J;

- *The Skew Right Unitors.* The natural transformation

$$\rho^{\mathbf{Rel}(A,B)}\colon \triangleright_J \circ \left(\mathrm{id} \times \mathbb{1}^{\mathbf{Rel}(A,B)}\right) \Longrightarrow \mathrm{id},$$

whose component

$$\rho^{\mathbf{Rel}(A,B)}_S\colon \underbrace{S \triangleright_J J}_{\stackrel{\text{def}}{=}\mathsf{Ran}_J(S)\diamond J} \subset S$$

at S is given by

$$\rho^{\mathbf{Rel}(A,B)}_S \stackrel{\text{def}}{=} \epsilon_R,$$

where $\epsilon\colon \mathsf{Ran}_J \diamond J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J^* \dashv \mathsf{Ran}_J$.

A.4 Right Relative Preorders

Let A and B be sets and let $J\colon A \nrightarrow B$ be a relation.

Definition A.4.1 ▶ Right J-Relative Preorders

A **right J-relative preorder from A to B** is equivalently:

- An $\mathbb{E}_1$-skew monoid in $(\mathrm{N}_\bullet(\mathbf{Rel}(A,A)), \triangleright_J, J)$;

- A skew monoid in $\left(\mathbf{Rel}(A,C), \rhd_J, J\right)$.

Remark A.4.2 ▶ Unwinding Definition A.4.1, I

In detail, a **right J-relative preorder** (R, μ_R, η_R) **from** A **to** B consists of

- *The Underlying Relation.* A relation
$$R\colon A \nrightarrow B,$$
called the **underlying relation of** (R, μ_R, η_R);
- *The Multiplication Inclusion.* An inclusion of relations
$$\mu_R\colon R \rhd_J R \subset R,$$
called the **multiplication of** (R, μ_R, η_R);
- *The Unit Inclusion.* An inclusion of relations
$$\eta_R\colon J \subset R,$$
called the **unit of** (R, μ_R, η_R).

Remark A.4.3 ▶ Unwinding Definition A.4.1, II

In other words, a **right J-relative preorder from** A **to** B is a relation $R\colon A \nrightarrow B$ from A to B satisfying the following conditions:

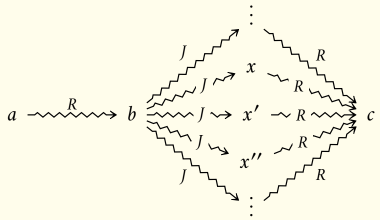
1. *J-Transitivity.* For each $a \in A$ and each $c \in B$, the following condition is satisfied:[1]

   (★) If there exists some $b \in B$ such that:
   - We have $a \sim_R b$;
   - For each $x \in A$, if $x \sim_J b$, then $x \sim_R c$;[2]

   then $a \sim_R c$.
2. *J-Unitality.* For each $a \in A$ and each $b \in B$, the following condition is satisfied:

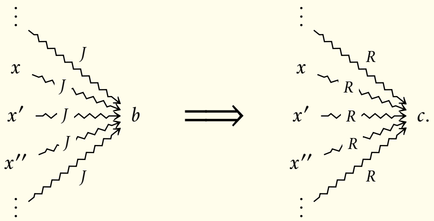
   (★) If $a \sim_J b$, then $a \sim_R b$.

[1] If we have



then $a \sim_R c$.

[2] Illustration:



B Other Chapters

Topology

Differential Geometry

Lie Groups and Lie Algebras

Homotopy Theory

Schemes

Morphisms of Schemes

Topics in Scheme Theory

Fundamental Groups of Schemes

Cohomology of Schemes

Group Schemes

References

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