

Posets

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INTRODUCTION

This chapter contains some material about posets and constructions with them. Notably, it contains:

- A basic discussion of posets, constructions with them, and co/limits inside posets ([Sections 1 to 4](#))
- A discussion of so-called *relative preorders* from a set X to a set Y . These are supposed to be an extension of the notion of a preorder $\leq_X : X \dashv\vdash X$ on a set X but where we allow the source and target of \leq_X to be entirely different sets.
The basic idea is that we may view preorders as precisely the monads in \mathbf{Rel} , so *relative preorders* are to be defined as *relative monads* in \mathbf{Rel} in the sense of [\[nLab23\]](#).
Thus, if you're interested in relative monads, you might like reading [Section A](#).

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1 Preorders and Partial Orders

1.1 Preorders

Let A be a set.

DEFINITION 1.1.1 ► PREORDERS

A **preorder on A** is equivalently:¹

- An \mathbb{E}_1 -monoid in $(\mathbf{N}_\bullet(\mathbf{Rel}(A, A)), \chi_A)$;
- A monoid in $(\mathbf{Rel}(A, A), \chi_A)$.

¹Note that since $\mathbf{Rel}(A, A)$ is posetal, being a preorder is a property of a relation, instead of a structure.

REMARK 1.1.2 ► UNWINDING DEFINITION 1.1.1

In detail, a relation R on A is a **preorder** if there exists

- *The Multiplication Inclusion.* An inclusion

$$\mu_R: R \diamond R \subset R$$

of relations in $\mathbf{Rel}(A, A)$, i.e. if, for each $a, c \in A$, we have:

- (★) If $a \sim_R b$ and $b \sim_R c$, then $a \sim_R c$.

- *The Unit Inclusion.* An inclusion

$$\eta_R: \chi_A \subset R$$

of relations in $\mathbf{Rel}(A, A)$, i.e. if, for each $a \in A$, we have $a \sim_R a$.

DEFINITION 1.1.3 ► THE PO/SET OF PREORDERS ON A SET

Let A be a set.

1. The **set of preorders on A** is the subset $\mathbf{POrd}(A, A)$ of $\mathbf{Rel}(A, A)$ spanned by the preorders.
2. The **poset of preorders on A** is the subposet $\mathbf{POrd}(A, A)$ of $\mathbf{Rel}(A, A)$ spanned by the preorders.

1.2 The Preorder Associated to a Relation

Let R be a relation on A .

DEFINITION 1.2.1 ► THE PREORDER ASSOCIATED TO A RELATION

The **preorder associated to R** is the preorder \sim_R^{pord} ¹ satisfying the following universal property:²

- (UP) Given another preorder \sim_S on A such that $R \subset S$, there exists an inclusion $\sim_R^{\text{pord}} \subset \sim_S$.

¹Further Notation: Also written R^{pord} .

²Slogan: The preorder associated to R is the smallest preorder containing R .

CONSTRUCTION 1.2.2 ► THE PREORDER ASSOCIATED TO A RELATION

Concretely, \sim_R^{pord} is the free monoid on R in $(\mathbf{Rel}(A, A), \diamond, \chi_A)^1$, being given by

$$\begin{aligned} R^{\text{pord}} &\stackrel{\text{def}}{=} \prod_{n=0}^{\infty} R^{\diamond n} \\ &\stackrel{\text{def}}{=} \Delta_A \cup \bigcup_{n=1}^{\infty} R^{\diamond n} \\ &\stackrel{\text{def}}{=} \left\{ (a, b) \in A \times B \mid \begin{array}{l} \text{we have } a = b \text{ or there exist} \\ (x_1, \dots, x_n) \in R^{\times n} \text{ such that} \\ a \sim_R x_1 \sim_R \dots \sim_R x_n \sim_R b \end{array} \right\}. \end{aligned}$$

¹Or, equivalently, the free \mathbb{B}_1 -monoid on R in $(\mathbf{N}_\bullet(\mathbf{Rel}(A, A)), \diamond, \chi_A)$.

PROOF 1.2.3 ► PROOF OF CONSTRUCTION 1.2.2

Clear. 

PROPOSITION 1.2.4 ► PROPERTIES OF THE PREORDER ASSOCIATED TO RELATION

Let R be a relation on A .

1. *Adjointness.* We have an adjunction

$$\left((-)^{\text{pord}} \dashv \overset{\circlearrowleft}{\sim} \right): \mathbf{Rel}(A, A) \begin{array}{c} \xrightarrow{(-)^{\text{pord}}} \\ \perp \\ \xleftarrow{\overset{\circlearrowleft}{\sim}} \end{array} \mathbf{POrd}(A, A),$$

witnessed by a bijection of sets

$$\mathbf{POrd}(\sim_R^{\text{pord}}, \sim_S) \cong \mathbf{Rel}(\sim_R, \sim_S),$$

natural in $\sim_R \in \text{Obj}(\mathbf{POrd}(A, A))$ and $\sim_S \in \text{Obj}(\mathbf{Rel}(A, A))$.

2. *The Associated Preorder of a Preorder.* If R is partial order, then $R^{\text{pord}} = R$.
3. *Idempotency.* We have

$$\left(R^{\text{pord}} \right)^{\text{pord}} = R^{\text{pord}}.$$

PROOF 1.2.5 ► PROOF OF PROPOSITION 1.2.4

Item 1: Adjointness

This is a rephrasing of the universal property of the preorder associated to a relation, stated in [Definition 1.2.1](#).

Item 2: The Associated Preorder of a Preorder

Clear.

Item 3: Idempotency

Clear. 

1.3 Partial Orders

Let X be a set.

DEFINITION 1.3.1 ► PARTIAL ORDERS

A **partial order** on X is a preorder \leq_X on X satisfying the following condition:

- (★) For each $x, y \in X$, if $x \leq_X y$ and $y \leq_X x$, then $x = y$.

REMARK 1.3.2 ► UNWINDING DEFINITION 1.3.1

In detail, a **partial order** is a relation $\leq_X : X \dashrightarrow X$ on X satisfying the following conditions:

1. *Reflexivity*. For each $x \in X$, we have $x \leq_X x$.
2. *Transitivity*. For each $x, y, z \in X$, if $x \leq_X y$ and $y \leq_X z$, then $x \leq_X z$.
3. *Antisymmetry*. For each $x, y \in X$, if $x \leq_X y$ and $y \leq_X x$, then $x = y$.

DEFINITION 1.3.3 ► THE PO/SET OF PARTIAL ORDERS ON A SET

Let X be a set.

1. The **set of partial orders relations on X** is the subset $\text{PartOrd}(X, X)$ of $\text{Rel}(X, X)$ spanned by the partial orders.
2. The **poset of partial orders relations on X** is the subposet $\text{PartOrd}(X, X)$ of $\text{Rel}(X, X)$ spanned by the partial orders.

1.4 The Partial Orders Associated to a Relation

Let R be a relation on X .

DEFINITION 1.4.1 ► THE PARTIAL ORDER ASSOCIATED TO A RELATION

The **partial order associated to R** is the partial order \sim_R^{ptord} ¹ satisfying the following universal property:²

(UP) Given another partial order \sim_S on X such that $R \subset S$, there exists an inclusion $\sim_R^{\text{ptord}} \subset \sim_S$.

¹Further Notation: Also written R^{ptord} .

²Slogan: The partial order associated to R is the smallest partial order containing R .

CONSTRUCTION 1.4.2 ► THE PARTIAL ORDER ASSOCIATED TO A RELATION

Concretely, \sim_R^{ptord} is the partial order on X defined by

$$\begin{aligned} R^{\text{ptord}} &\stackrel{\text{def}}{=} \left(R^{\text{antisymm}} \right)^{\text{ptord}} \\ &\cong (R/\sim)^{\text{ptord}} \\ &\stackrel{\text{def}}{=} \Delta_A \cup \bigcup_{n=1}^{\infty} (R/\sim)^{\circ n}, \end{aligned}$$

where \sim is the equivalence relation on R obtained by declaring $a \sim b$ iff $a \sim_R b$ and $b \sim_R a$.

PROOF 1.4.3 ► PROOF OF CONSTRUCTION 1.4.2

Clear. 

PROPOSITION 1.4.4 ► PROPERTIES OF THE PARTIAL ORDER ASSOCIATED TO RELATION

Let R be a relation on X .

1. *Adjointness.* We have an adjunction

$$\left((-)^{\text{ptord}} \dashv \text{忘} \right): \mathbf{Rel}(X, X) \begin{matrix} \xrightarrow{(-)^{\text{ptord}}} \\ \perp \\ \xleftarrow{\text{忘}} \end{matrix} \mathbf{PartOrd}(X, X),$$

witnessed by a bijection of sets

$$\mathbf{PartOrd}(\sim_R^{\text{ptord}}, \sim_S) \cong \mathbf{Rel}(\sim_R, \sim_S),$$

natural in $\sim_R \in \text{Obj}(\mathbf{PartOrd}(X, X))$ and $\sim_S \in \text{Obj}(\mathbf{Rel}(X, X))$.

2. *The Associated Preorder of a Preorder.* If R is partial order, then $R^{\text{ptord}} = R$.
3. *Idempotency.* We have

$$(R^{\text{ptord}})^{\text{ptord}} = R^{\text{ptord}}.$$

PROOF 1.4.5 ► PROOF OF PROPOSITION 1.4.4

Item 1: Adjointness

This is a rephrasing of the universal property of the partial order associated to a relation, stated in [Definition 1.4.1](#).

Item 2: The Associated Preorder of a Preorder

Clear.

Item 3: Idempotency

Clear. 

1.5 Total Orders

Let X be a set.

DEFINITION 1.5.1 ► TOTAL ORDERS

A **total order on X** is a partial order \leq_X on X satisfying the following condition:

- (★) For each $x, y \in X$, we have either $x \leq_X y$ or $y \leq_X x$.

2 Posets

2.1 Foundations

DEFINITION 2.1.1 ► POSETS

A **poset** (X, \leq_X) consists of

- *The Underlying Set.* A set X , called the **underlying set of** (X, \leq_X) ;
- *The Partial Order.* A partial order

$$\leq_X: X \times X \rightarrow \{\text{true}, \text{false}\}$$

on X , called the **partial order of** (X, \leq_X) .

EXAMPLE 2.1.2 ► POWERSETS

Given a set X , the pair $(\mathcal{P}(X), \subset)$ is a poset, as is $(\mathcal{P}(X), \supset)$.

DEFINITION 2.1.3 ► THE POSETAL CHARACTERISTIC RELATION OF A POSET

The **posetal characteristic relation** of a poset (X, \leq) is the relation

$$\chi_{(X, \leq)}^{\text{Pos}}: X \times X \rightarrow \{\text{true}, \text{false}\}$$

on X defined by¹

$$\chi_{(X, \leq)}^{\text{Pos}}(x, y) \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } x \leq y, \\ \text{false} & \text{if } x \not\leq y \end{cases}$$

for each $x, y \in X$.

¹In other words, $\chi_{(X, \leq)}^{\text{Pos}}$ is just the Hom of the posetal category associated to (X, \leq) , defined by

$$\text{Hom}_{(X, \leq)}(x, y) \stackrel{\text{def}}{=} \begin{cases} \text{pt} & \text{if } x \leq y, \\ \emptyset & \text{if } x \not\leq y. \end{cases}$$

but one level lower in enrichment, as $\chi_{(X, \leq)}^{\text{Pos}}$ takes values in $\{\text{true}, \text{false}\}$, instead of in $\{\text{pt}, \emptyset\} \subset \text{Sets}$.

2.2 Morphisms of Posets

Let (X, \leq_X) and (Y, \leq_Y) be posets.

DEFINITION 2.2.1 ► MORPHISMS OF POSETS

A **morphism of posets from** (X, \leq_X) **to** (Y, \leq_Y) ¹ is a function $f: X \rightarrow Y$ satisfying the following condition:

(★) *Monotonicity.* For each $x, y \in X$, if $x \leq_X y$, then $f(x) \leq_Y f(y)$.

¹*Further Terminology:* Also called a **monotone function**.

2.3 Ideals of Posets

Let (X, \leq_X) be a poset.

DEFINITION 2.3.1 ► IDEALS OF POSETS

An **ideal of** (X, \leq_X) is a subset I of X satisfying the following conditions:

1. *Non-Emptiness.* We have $I \neq \emptyset$.
2. *Upward-Directedness.* For each $x, y \in I$, there exists some $c_{x,y} \in I$ such that:
 - We have $x \leq_X c_{x,y}$.
 - We have $y \leq_X c_{x,y}$.
3. *Downward-Closedness.* For each $x, y \in I$, if:
 - We have $y \in I$;
 - We have $x \leq_X y$;
 then $x \in I$.

REMARK 2.3.2 ► ALTERNATIVE AXIOMS FOR IDEALS OF LATTICES

If (X, \leq_X) is a lattice, then $I \subset X$ is an ideal of (X, \leq_X) iff the following conditions are satisfied:

1. *Containment of the Bottom Element.* We have $\perp \in I$.
2. *Closure Under Binary Joins.* If $x, y \in I$, then $x \vee y \in I$.
3. *Closure Under Binary Joins With Elements of X .* If $a \in X$ and $x \in I$, then $a \vee x \in I$.

DEFINITION 2.3.3 ► PROPER IDEALS

An ideal I of (X, \leq_X) is **proper** if $I \neq X$.

DEFINITION 2.3.4 ► PRIME IDEALS

An ideal I of (X, \leq_X) is **prime** if the following conditions are satisfied:

1. *Properness.* The ideal I is proper.
2. *Primality.* For each $x, y \in I$, if $x \vee y \in I$, then $x \in I$ or $y \in I$.

DEFINITION 2.3.5 ► COMPLETELY PRIME IDEALS

An ideal I of a lattice (X, \leq_X) is **completely prime** if the following conditions are satisfied:

1. *Properness.* The ideal I is proper.
2. *Infinitary Primality.* For each $\{x_i\}_{i \in J} \in \mathcal{P}(I)$, if $\bigvee_{i \in J} x_i \in I$, then there exists some $i \in I$ such that $x_i \in I$.

DEFINITION 2.3.6 ► MAXIMAL IDEALS

An ideal I of (X, \leq_X) is **maximal** if the following conditions are satisfied:

1. *Properness.* The ideal I is proper.
2. *Maximality.* Given another ideal J of X , if $I \subset J$, then $J = X$.

2.4 Filters on Posets**2.4.1 Foundations**

Let (X, \leq_X) be a poset.

DEFINITION 2.4.1 ► FILTERS ON POSETS

A **filter on** (X, \leq_X) is a subset F of X satisfying the following conditions:

1. *Non-Emptiness.* We have $F \neq \emptyset$.
2. *Upward-Closedness.* For each $x, y \in X$, if:

- We have $x \in F$;
- We have $x \leq_X y$;

then $y \in F$.

3. *Downward-Directedness.* For each $x, y \in F$, there exists some $c_{x,y} \in F$ such that:

- We have $c_{x,y} \leq_X x$;
- We have $c_{x,y} \leq_X y$.

REMARK 2.4.2 ► ALTERNATIVE AXIOMS FOR FILTERS ON LATTICES

If (X, \leq_X) is a lattice, then $F \subset X$ is a filter on (X, \leq_X) iff the following conditions are satisfied:¹

1. *Containment of the Top Element.* We have $\top \in F$.
2. *Closure Under Binary Meets.* If $x, y \in F$, then $x \wedge y \in F$.
3. *Closure Under Binary Joins With Elements of X .* If $a \in X$ and $x \in F$, then $a \vee x \in F$.

¹These conditions are equivalent to the statement that $\chi_F: X \rightarrow \{\text{true}, \text{false}\}$ is a morphism of meet-semilattices.

2.4.2 Proper Filters

Let (X, \leq_X) be a poset.

DEFINITION 2.4.3 ► PROPER FILTERS

A filter F on X is **proper** if $F \neq X$.¹

¹*Further Terminology:* The filter X on X is called the **improper filter**.

2.4.3 Prime Filters

Let (X, \leq_X) be a lattice.

DEFINITION 2.4.4 ► PRIME FILTERS

A filter F on X is **prime** if $X \setminus F$ is an ideal of X .

REMARK 2.4.5 ► UNWINDING DEFINITION 2.4.4

That is, F is **prime** if the following conditions are satisfied:

- *Properness.* We have $\perp \notin F$.
- *Primality.* For each $x, y \in X$, if $x \vee y \in F$, then $x \in F$ or $y \in F$.

2.4.4 Completely Prime Filters

Let (X, \leq_X) be a lattice.

DEFINITION 2.4.6 ► COMPLETELY PRIME FILTERS

A filter F on X is **completely prime** if the following conditions are satisfied:

- *Properness.* We have $\perp \notin F$.
- *Primality.* For each $\{x_i\}_{i \in I} \in \mathcal{PC}(X)$, if $\bigvee_{i \in I} x_i \in F$, then there exists some $i \in I$ such that $x_i \in F$.

2.4.5 Ultrafilters**3 Constructions With Posets****3.1 The Dual of a Poset**

Let (X, \leq_X) be a poset.

DEFINITION 3.1.1 ► THE DUAL OF A POSET

The **dual of** (X, \leq_X) is the poset $(X^{\text{op}}, \leq_{X^{\text{op}}})$ consisting of

- *The Underlying Set.* The set X^{op} defined by

$$X^{\text{op}} \stackrel{\text{def}}{=} X;$$

- *The Partial Order.* The partial order

$$\leq_{X^{\text{op}}}: X^{\text{op}} \times X^{\text{op}} \rightarrow \{\text{true}, \text{false}\}$$

on X^{op} defined by

$$x \leq_{X^{\text{op}}} y \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } y \leq_X x, \\ \text{false} & \text{otherwise.} \end{cases}$$

EXAMPLE 3.1.2 ► DUAL OF POWERSETS

Let X be a set. The dual of $(\mathcal{P}(X), \subset)$ is $(\mathcal{P}(X), \supset)$.

3.2 Products of Posets

Let (X, \leq_X) and (Y, \leq_Y) be posets.

DEFINITION 3.2.1 ► PRODUCTS OF POSETS

The **product** of (X, \leq_X) and (Y, \leq_Y) is the poset $(X \times Y, \leq_{X \times Y})$ consisting of

- *The Underlying Set.* The Cartesian product $X \times Y$ of X and Y ;
- *The Partial Order.* The partial order

$$\leq_{X \times Y}: (X \times Y) \times (X \times Y) \rightarrow \{\text{true}, \text{false}\}$$

on $X \times Y$ defined by

$$(a, b) \leq_{X \times Y} (x, y) \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } a \leq_X x \text{ and } b \leq_Y y, \\ \text{false} & \text{otherwise.} \end{cases}$$

3.3 Coproducts of Posets

Let (X, \leq_X) and (Y, \leq_Y) be posets.

DEFINITION 3.3.1 ► COPRODUCTS OF POSETS

The **coproduct** of (X, \leq_X) and (Y, \leq_Y) is the poset $(X \amalg Y, \leq_{X \amalg Y})$ consisting of

- *The Underlying Set.* The disjoint union $X \amalg Y$ of X and Y ;

- *The Partial Order.* The partial order

$$\leq_{X \amalg Y} : (X \amalg Y) \times (X \amalg Y) \rightarrow \{\text{true}, \text{false}\}$$

on $X \amalg Y$ defined by

$$x \leq_{X \amalg Y} y \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } x, y \in X \text{ and } x \leq_X y, \\ \text{true} & \text{if } x, y \in Y \text{ and } x \leq_Y y, \\ \text{false} & \text{otherwise.} \end{cases}$$

3.4 The Tensor Product of Posets

3.4.1 Bilinear Morphisms of Posets

Let (X, \leq_X) , (Y, \leq_Y) , and (Z, \leq_Z) be posets.

DEFINITION 3.4.1 ► BILINEAR MORPHISMS OF POSETS

A **bilinear morphism of posets** from $(X \times Y, \leq_{X \times Y})$ to (Z, \leq_Z) is a function $f: X \times Y \rightarrow Z$ satisfying the following conditions:

- For each $x, y \in X$ and each $z \in Y$, if $x \leq_X y$, then $f(x, z) \leq_Z f(y, z)$.
- For each $x \in X$ and each $y, z \in Y$, if $y \leq_Y z$, then $f(x, y) \leq_Z f(x, z)$.

3.4.2 The Tensor Product of Posets

Let (X, \leq_X) and (Y, \leq_Y) be posets.

DEFINITION 3.4.2 ► THE TENSOR PRODUCT OF POSETS

The **tensor product** of (X, \leq_X) and (Y, \leq_Y) is the pair $(X \boxtimes Y, \iota)$ consisting of

- The poset $X \boxtimes Y$;
- The bilinear morphism of posets $\iota: X \times Y \rightarrow X \boxtimes Y$;

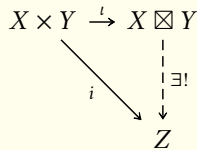
satisfying the following universal property:

(UP) Given another pair (Z, i) consisting of

- A poset Z ;

· A bilinear morphism of posets $i: X \times Y \rightarrow Z$;

there exists a unique morphism of posets $X \boxtimes Y \xrightarrow{\exists!} Z$ making the diagram



commute.

CONSTRUCTION 3.4.3 ► TENSOR PRODUCTS OF POSETS

Concretely, the **tensor product of X and Y** is the poset $X \boxtimes Y$ defined by

$$X \boxtimes Y \stackrel{\text{def}}{=} X_{\text{cat}} \times Y_{\text{disc}} \coprod_{X_{\text{disc}} \times Y_{\text{disc}}} X_{\text{disc}} \times Y_{\text{cat}},$$

$$\begin{array}{ccc}
 X \boxtimes Y & \longleftarrow & X_{\text{cat}} \times Y_{\text{disc}} \\
 \uparrow & \lrcorner & \uparrow \\
 X_{\text{disc}} \times Y_{\text{cat}} & \longleftarrow & X_{\text{disc}} \times Y_{\text{disc}}
 \end{array}$$

3.5 Internal Homs

Let (X, \leq_X) and (Y, \leq_Y) be posets.

DEFINITION 3.5.1 ► INTERNAL HOMS OF POSETS

The **internal Hom of posets from (X, \leq_X) to (Y, \leq_Y)** is the poset $\mathbf{Pos}((X, \leq_X), (Y, \leq_Y))^1$ consisting of

- *The Underlying Set.* The set $\mathbf{Pos}((X, \leq_X), (Y, \leq_Y))$ of morphisms of posets from (X, \leq_X) to (Y, \leq_Y) ;
- *The Partial Order.* The partial order

$$\leq_{\mathbf{Pos}(X, Y)} : \mathbf{Pos}(X, Y) \times \mathbf{Pos}(X, Y) \rightarrow \{\text{true}, \text{false}\}$$

on $\mathbf{Pos}(X, Y)$ defined by²

$$f \leq g \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } f(x) \leq g(x) \text{ for each } x \in X, \\ \text{false} & \text{otherwise} \end{cases}$$

for each $f, g \in \text{Pos}(f, g)$.

¹*Further Notation:* Also written simply $\text{Pos}(X, Y)$.

²*Further Terminology:* Due to its definition, $\leq_{\text{Pos}(X, Y)}$ is called the **pointwise partial order** on $\text{Pos}(X, Y)$.

4 Co/Limits in Posets

4.1 Initial Elements

Let (X, \leq_X) be a poset.

DEFINITION 4.1.1 ► THE INITIAL ELEMENT OF A POSET

The **initial element** of (X, \leq_X) ¹ is, if it exists, the element \perp of X satisfying the following condition:

(★) For each $x \in X$, we have $\perp \leq_X x$.

¹*Further Terminology:* Also called the **bottom element** of (X, \leq_X) .

EXAMPLE 4.1.2 ► THE INITIAL ELEMENT OF A POWERSET

Let X be a set. The initial element of $(\mathcal{P}(X), \subset)$ is given by \emptyset .

4.2 Final Elements

Let (X, \leq_X) be a poset.

DEFINITION 4.2.1 ► THE FINAL ELEMENT OF A POSET

The **final element** of (X, \leq_X) ¹ is, if it exists, the element \top of X satisfying the following condition:

(★) For each $x \in X$, we have $x \leq_X \top$.

¹*Further Terminology:* Also called the **top element** of (X, \leq_X) .

EXAMPLE 4.2.2 ► THE FINAL ELEMENT OF A POWERSET

Let X be a set. The final element of $(\mathcal{P}(X), \subset)$ is given by X .

4.3 Binary Joins

Let (X, \leq_X) be a poset and let $x, y \in X$.

DEFINITION 4.3.1 ► BINARY JOINS IN A POSET

The **binary join of x and y** in (X, \leq_X) is, if it exists, the element $x \vee y$ of X satisfying the following conditions:

1. We have $x \leq_X x \vee y$ and $y \leq_X x \vee y$.
2. For each $s \in X$, if $x \leq_X s$ and $y \leq_X s$, then $x \vee y \leq_X s$.

EXAMPLE 4.3.2 ► BINARY JOINS IN POWERSETS

Let X be a set. The binary join of U and V in $(\mathcal{P}(X), \subset)$ is given by $U \cup V$.

4.4 Joins of Families

Let (X, \leq_X) be a poset and let $\{x_i\}_{i \in I}$ be a family of elements of X .

DEFINITION 4.4.1 ► JOINS OF FAMILIES OF ELEMENTS IN A POSET

The **join of $\{x_i\}_{i \in I}$** in (X, \leq_X) is, if it exists, the element $\bigvee_{i \in I} x_i$ of X satisfying the following conditions:

1. For each $i \in I$, we have $x_i \leq_X \bigvee_{i \in I} x_i$.
2. For each $s \in X$, the following condition is satisfied:
 (★) If, for each $i \in I$, we have $x_i \leq_X s$, then $\bigvee_{i \in I} x_i \leq_X s$.

EXAMPLE 4.4.2 ► JOINS OF EMPTY FAMILIES

The meet $\bigvee_{i \in \emptyset} x_i$ of the empty family is given by (if it exists) the bottom element \perp of (X, \leq_X) .

EXAMPLE 4.4.3 ► JOINS OF FAMILIES IN POWERSETS

Let X be a set. The join of a family $\{U_i\}_{i \in I}$ in $(\mathcal{P}(X), \subset)$ is given by $\bigcup_{i \in I} U_i$.

4.5 Binary Meets

Let (X, \leq_X) be a poset and let $x, y \in X$.

DEFINITION 4.5.1 ► BINARY MEETS IN A POSET

The **binary meet of x and y in (X, \leq_X)** is, if it exists, the element $x \wedge y$ of X satisfying the following conditions:

1. We have $x \wedge y \leq_X x$ and $x \wedge y \leq_X y$.
2. For each $a \in X$, if $a \leq_X x$ and $a \leq_X y$, then $a \leq_X x \wedge y$.

EXAMPLE 4.5.2 ► BINARY MEETS IN POWERSETS

Let X be a set. The binary meet of U and V in $(\mathcal{P}(X), \subset)$ is given by $U \cap V$.

4.6 Meets of Families

Let (X, \leq_X) be a poset and let $\{x_i\}_{i \in I}$ be a family of elements of X .

DEFINITION 4.6.1 ► MEETS OF FAMILIES OF ELEMENTS IN A POSET

The **meet of $\{x_i\}_{i \in I}$ in (X, \leq_X)** is, if it exists, the element $\bigwedge_{i \in I} x_i$ of X satisfying the following conditions:

1. For each $i \in I$, we have $\bigwedge_{i \in I} x_i \leq_X x_i$.
2. For each $s \in X$, the following condition is satisfied:
 - (★) If, for each $i \in I$, we have $s \leq_X x_i$, then $s \leq_X \bigwedge_{i \in I} x_i$.

EXAMPLE 4.6.2 ► MEETS OF EMPTY FAMILIES

The meet $\bigwedge_{i \in \emptyset} x_i$ of the empty family is given by (if it exists) the top element \top of (X, \leq_X) .

EXAMPLE 4.6.3 ► MEETS OF FAMILIES IN POWERSETS

Let X be a set. The meet of a family $\{U_i\}_{i \in I}$ in $(\mathcal{P}(X), \subset)$ is given by $\bigcap_{i \in I} U_i$.

4.7 Lattices

DEFINITION 4.7.1 ► LATTICES

Let (X, \leq_X) be a poset.

1. The poset (X, \leq_X) is a **join-semilattice** if it has a bottom element and binary joins.¹
2. The poset (X, \leq_X) is a **meet-semilattice** if it has a top element and binary meets.²
3. The poset (X, \leq_X) is a **suplattice** if it has joins of arbitrary families.
4. The poset (X, \leq_X) is an **inflat** if it has meets of arbitrary families.
5. The poset (X, \leq_X) is a **lattice** if it is both a join-semilattice and a meet-semilattice.
6. The poset (X, \leq_X) is a **complete lattice** if it is both a lattice and an inflat.
7. The poset (X, \leq_X) is a **cocomplete lattice** if it is both a lattice and a suplattice.
8. The poset (X, \leq_X) is a **bicomplete lattice** if it is both a complete lattice and a cocomplete lattice.

¹This is equivalent to having joins of finite families.

²This is equivalent to having meets of finite families.

Appendices

A Relative Preorders

A.1 The Left Skew Monoidal Structure on $\mathbf{Rel}(A, B)$

Let A and B be sets and let $J : A \dashrightarrow B$ be a relation.

DEFINITION A.1.1 ► THE LEFT J -SKEW MONOIDAL STRUCTURE ON $\mathbf{REL}(A, B)$

The **left J -skew monoidal category of functors from A to B** is the left skew monoidal category $(\mathbf{Rel}(A, B), \triangleleft_J, J)$ consisting of

- *The Underlying Category.* The category $\mathbf{Rel}(A, B)$ of relations from A to B ;
- *The Skew Monoidal Product.* The functor

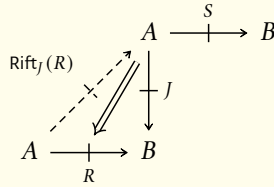
$$\triangleleft_J: \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \rightarrow \mathbf{Rel}(A, B)$$

from $\mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B)$ to $\mathbf{Rel}(A, B)$, called the **left J -skew monoidal product of relations from A to B** , where

- *Action on Objects.* For each $R, S \in \text{Obj}(\mathbf{Rel}(A, B))$, we have

$$S \triangleleft_J R \stackrel{\text{def}}{=} S \diamond \text{Rift}_J(R),$$

where $S \diamond \text{Rift}_J(R)$ is the composition



in \mathbf{Rel} ;

- *Action on Morphisms.* For each $R, S, R', S' \in \text{Obj}(\mathbf{Rel}(A, B))$, the action on Hom-sets

$$(\triangleleft_J)_{(G,F),(G',F')} : \text{Hom}_{\mathbf{Rel}(A,B)}(S, S') \times \text{Hom}_{\mathbf{Rel}(A,B)}(R, R') \rightarrow \text{Hom}_{\mathbf{Rel}(A,B)}(S \triangleleft_J R, S' \triangleleft_J R')$$

of \triangleleft_J at $((R, S), (R', S'))$ is defined by

$$\beta \triangleleft_J \alpha \stackrel{\text{def}}{=} \beta \diamond \text{Rift}_J(\alpha),$$

for each $\beta \in \text{Hom}_{\mathbf{Rel}(A,B)}(S, S')$ and each $\alpha \in \text{Hom}_{\mathbf{Rel}(A,B)}(R, R')$;

- *The Skew Monoidal Unit.* The functor

$$\mathbb{K}^{\mathbf{Rel}(A,B)}: \text{pt} \rightarrow \mathbf{Rel}(A, B)$$

defined by

$$\mathbb{K}^{\mathbf{Rel}(A,B)} \stackrel{\text{def}}{=} J;$$

- *The Skew Associators.* The natural transformation

$$\alpha^{\mathbf{Rel}(A,B)} : \triangleleft_J \circ (\triangleleft_J \times \text{id}) \Longrightarrow \triangleleft_J \circ (\text{id} \times \triangleleft_J),$$

whose component

$$\alpha_{T,S,R}^{\mathbf{Rel}(A,B)} : \underbrace{(T \triangleleft_J S) \triangleleft_J R}_{\stackrel{\text{def}}{=} T \diamond \text{Rift}_J(S) \diamond \text{Rift}_J(R)} \subset \underbrace{T \triangleleft_J (S \triangleleft_J R)}_{\stackrel{\text{def}}{=} T \diamond \text{Rift}_J(S \diamond \text{Rift}_J(R))}$$

at (T, S, R) is given by

$$\alpha_{T,S,R}^{\mathbf{Rel}(A,B)} \stackrel{\text{def}}{=} \text{id}_T \diamond \gamma,$$

where

$$\gamma : \text{Rift}_J(S) \diamond \text{Rift}_J(R) \subset \text{Rift}_J(S \diamond \text{Rift}_J(R))$$

is the inclusion adjunct to the inclusion

$$\underbrace{J \diamond \text{Rift}_J(S) \diamond \text{Rift}_J(R)}_{\stackrel{\text{def}}{=} J_* (\text{Rift}_J(S) \diamond \text{Rift}_J(R))} \xrightarrow{\epsilon_S * \text{id}_{\text{Rift}_J(R)}} \subset S \diamond \text{Rift}_J(R)$$

under the adjunction $J_* \dashv \text{Rift}_J$, where $\epsilon : J \diamond \text{Rift}_J \Longrightarrow \text{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J_* \dashv \text{Rift}_J$;

- *The Skew Left Unitors.* The natural transformation

$$\lambda^{\mathbf{Rel}(A,B)} : \triangleleft_J \circ (\mu^{\mathbf{Rel}(A,B)} \times \text{id}) \Longrightarrow \text{id},$$

whose component

$$\lambda_R^{\mathbf{Rel}(A,B)} : \underbrace{J \triangleleft_J R}_{\stackrel{\text{def}}{=} J \diamond \text{Rift}_J(R)} \subset R$$

at R is given by

$$\lambda_R^{\mathbf{Rel}(A,B)} \stackrel{\text{def}}{=} \epsilon_R,$$

where $\epsilon : J \diamond \text{Rift}_J \Longrightarrow \text{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J_* \dashv \text{Rift}_J$;

- *The Skew Right Unitors.* The natural transformation

$$\rho^{\mathbf{Rel}(A,B)} : \text{id} \Longrightarrow \triangleleft_J \circ \left(\text{id} \times \mathbb{K}^{\mathbf{Rel}(A,B)} \right),$$

whose component

$$\rho_R^{\mathbf{Rel}(A,B)} : R \subset \underbrace{R \triangleleft_J J}_{\stackrel{\text{def}}{=} R \circ \text{Rift}_J(J)}$$

at R is given by

$$\rho_R^{\mathbf{Rel}(A,B)} \stackrel{\text{def}}{=} \text{id}_R \star \sigma,$$

where $\sigma : \text{id}_A \Longrightarrow \text{Rift}_J(J)$ is the universal transformation included in the data of the right Kan lift $\text{Rift}_J(J)$.

A.2 Left Relative Preorders

Let A and B be sets and let $J : A \dashrightarrow B$ be a relation.

DEFINITION A.2.1 ► LEFT J -RELATIVE PREORDERS

A **left J -relative preorder from A to B** is equivalently:

- An \mathbb{E}_1 -skew monoid in $(\mathbf{N}_\bullet(\mathbf{Rel}(A, A)), \triangleleft_J, J)$;
- A skew monoid in $(\mathbf{Rel}(A, C), \triangleleft_J, J)$.

REMARK A.2.2 ► UNWINDING DEFINITION A.2.1, I

In detail, a **left J -relative preorder (R, μ_R, η_R) from A to B** consists of

- *The Underlying Relation.* A relation

$$R : A \dashrightarrow B,$$

called the **underlying relation of (R, μ_R, η_R)** ;

- *The Multiplication Inclusion.* An inclusion of relations

$$\mu_R : R \triangleleft_J R \subset R,$$

called the **multiplication of (R, μ_R, η_R)** ;

· *The Unit Inclusion.* An inclusion of relations

$$\eta_R: J \subset R,$$

called the **unit of** (R, μ_R, η_R) .

REMARK A.2.3 ► UNWINDING DEFINITION A.2.1, II

In other words, a **left J -relative preorder from A to B** is a relation $R: A \dashrightarrow B$ from A to B satisfying the following conditions:

1. *J -Transitivity.* For each $a \in A$ and each $c \in B$, the following condition is satisfied:¹

(★) If there exists some $b \in A$ such that:

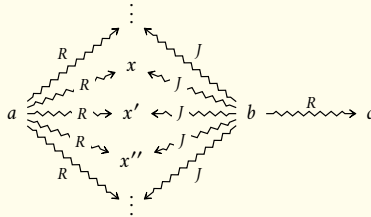
- For each $x \in B$, if $b \sim_J x$, then $a \sim_R x$;²
- We have $b \sim_R c$;

then $a \sim_R c$.

2. *J -Unitality.* For each $a \in A$ and each $b \in B$, the following condition is satisfied:

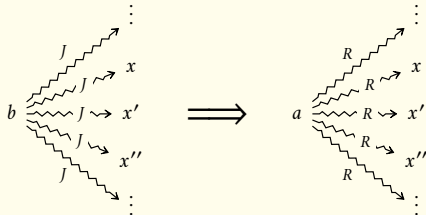
(★) If $a \sim_J b$, then $a \sim_R b$.

¹If we have



then $a \sim_R c$.

²Illustration:



A.3 The Right Skew Monoidal Structure on $\mathbf{Rel}(A, B)$

Let A and B be sets and let $J: A \dashrightarrow B$ be a relation.

DEFINITION A.3.1 ► THE RIGHT J -SKEW MONOIDAL STRUCTURE ON $\mathbf{REL}(A, B)$

The **right J -skew monoidal category of functors from A to B** is the right skew monoidal category consisting of

- *The Underlying Category.* The category $\mathbf{Rel}(A, B)$ of relations from A to B ;
- *The Skew Monoidal Product.* The functor

$$\triangleright_J: \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \rightarrow \mathbf{Rel}(A, B)$$

from $\mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B)$ to $\mathbf{Rel}(A, B)$, called the **right J -skew monoidal product of functors from A to B** , where

- *Action on Objects.* For each $R, S \in \text{Obj}(\mathbf{Rel}(A, B))$, we have

$$S \triangleright_J R \stackrel{\text{def}}{=} \text{Ran}_J(S) \diamond R,$$

where $\text{Ran}_J(S) \diamond R$ is the composition

$$\begin{array}{ccc}
 A & \xrightarrow{R} & B & \xrightarrow{\text{Ran}_J(S)} & B \\
 & & \uparrow J & \swarrow S & \nearrow \\
 & & A & &
 \end{array}$$

in \mathbf{Cats} ;

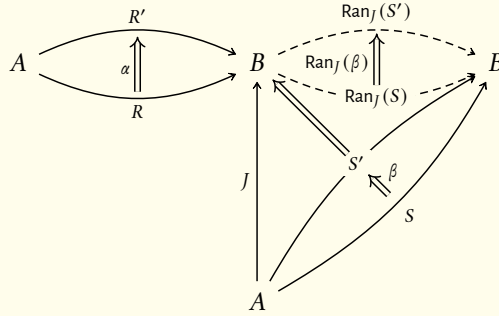
- *Action on Morphisms.* For each $R, S, R', S' \in \text{Obj}(\mathbf{Rel}(A, B))$, the action on Hom-sets

$$(\triangleright_J)_{(S,R),(S',R')} : \text{Nat}(S, S') \times \text{Nat}(R, R') \rightarrow \text{Nat}(S \triangleright_J R, S' \triangleright_J R')$$

of \triangleright_J at $((S, R), (S', R'))$ is defined by

$$\beta \triangleright_J \alpha \stackrel{\text{def}}{=} \text{Ran}_J(\beta) \diamond \alpha,$$

where $\text{Ran}_J(\beta) \diamond \alpha$ is the horizontal composition



in \mathbf{Cats} ;

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defined by

$$\mathbb{K}^{\mathbf{Rel}(A,B)} \stackrel{\text{def}}{=} J;$$

- *The Skew Associators.* The natural transformation

$$\alpha^{\mathbf{Rel}(A,B)} : \triangleright_J \circ (\text{id} \times \triangleright_J) \Longrightarrow \triangleright_J \circ (\triangleright_J \times \text{id}),$$

whose component

$$\alpha_{T,S,R}^{\mathbf{Rel}(A,B)} : \underbrace{T \triangleright_J (S \triangleright_J R)}_{\stackrel{\text{def}}{=} \text{Ran}_J(T) \diamond (\text{Ran}_J(S) \diamond R)} \subset \underbrace{(T \triangleright_J S) \triangleright_J R}_{\stackrel{\text{def}}{=} \text{Ran}_J(\text{Ran}_J(T) \diamond S) \diamond R}$$

at (T, S, R) is given by

$$\alpha_{T,S,R}^{\mathbf{Rel}(A,B)} \stackrel{\text{def}}{=} \gamma \diamond \text{id}_R,$$

where

$$\gamma : \text{Ran}_J(T) \diamond \text{Ran}_J(S) \subset \text{Ran}_J(\text{Ran}_J(T) \diamond S)$$

is the inclusion adjunct to the inclusion

$$\underbrace{\text{Ran}_J(T) \diamond \text{Ran}_J(S) \diamond J}_{\stackrel{\text{def}}{=} J^*(\text{Ran}_J(T) \diamond \text{Ran}_J(S))} \xrightarrow{\text{id}_{\text{Ran}_J(T)} \circ \varepsilon_S} \text{Ran}_J(T) \diamond S$$

under the adjunction $J^* \dashv \text{Ran}_J$, where $\varepsilon : \text{Ran}_J \diamond J \Longrightarrow \text{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J^* \dashv \text{Ran}_J$;

- *The Skew Left Unitors.* The natural transformation

$$\lambda^A : \text{id} \Longrightarrow \triangleright_J \circ \left(\mathbb{K}^{\mathbf{Rel}(A,B)} \times \text{id} \right),$$

whose component

$$\lambda_R^{\mathbf{Rel}(A,B)} : R \subset \underbrace{J \triangleright_J R}_{\stackrel{\text{def}}{=} \text{Ran}_J(J) \diamond R}$$

at R is given by

$$\lambda_R^{\mathbf{Rel}(A,B)} \stackrel{\text{def}}{=} \sigma \diamond \text{id}_R,$$

where $\sigma : \text{id}_B \Longrightarrow \text{Ran}_J(J)$ is the unit of the codensity monad of J ;

- *The Skew Right Unitors.* The natural transformation

$$\rho^{\mathbf{Rel}(A,B)} : \triangleright_J \circ \left(\text{id} \times \mathbb{K}^{\mathbf{Rel}(A,B)} \right) \Longrightarrow \text{id},$$

whose component

$$\rho_S^{\mathbf{Rel}(A,B)} : \underbrace{S \triangleright_J J}_{\stackrel{\text{def}}{=} \text{Ran}_J(S) \diamond J} \subset S$$

at S is given by

$$\rho_S^{\mathbf{Rel}(A,B)} \stackrel{\text{def}}{=} \varepsilon_R,$$

where $\varepsilon : \text{Ran}_J \diamond J \Longrightarrow \text{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J^* \dashv \text{Ran}_J$.

A.4 Right Relative Preorders

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- A skew monoid in $(\mathbf{Rel}(A, C), \triangleright_J, J)$.

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called the **underlying relation of** (R, μ_R, η_R) ;

- *The Multiplication Inclusion.* An inclusion of relations

$$\mu_R: R \triangleright_J R \subset R,$$

called the **multiplication of** (R, μ_R, η_R) ;

- *The Unit Inclusion.* An inclusion of relations

$$\eta_R: J \subset R,$$

called the **unit of** (R, μ_R, η_R) .

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1. *J -Transitivity.* For each $a \in A$ and each $c \in B$, the following condition is satisfied:¹

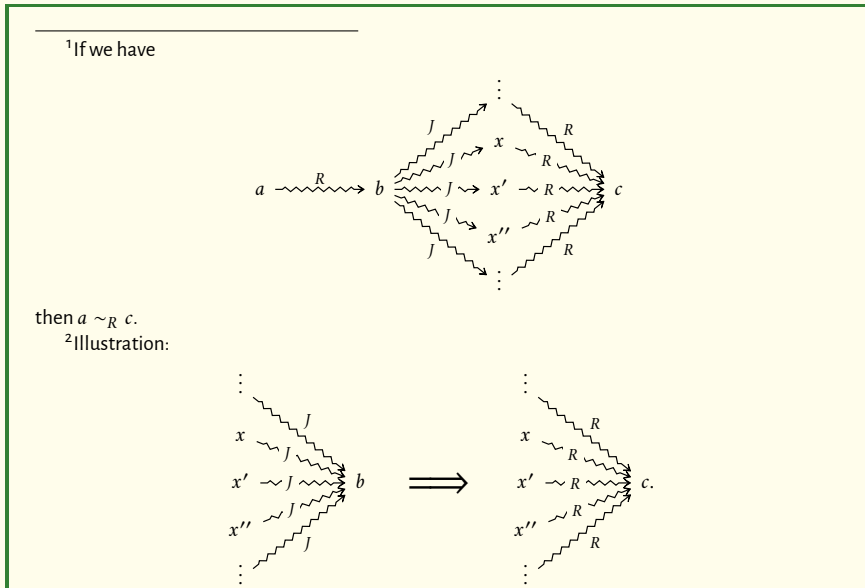
(★) If there exists some $b \in B$ such that:

- We have $a \sim_R b$;
- For each $x \in A$, if $x \sim_J b$, then $x \sim_R c$;²

then $a \sim_R c$.

2. *J -Unitality.* For each $a \in A$ and each $b \in B$, the following condition is satisfied:

(★) If $a \sim_J b$, then $a \sim_R b$.



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Logic and Model Theory

1. Logic
2. Model Theory

Type Theory

3. Type Theory
4. Homotopy Type Theory

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6. Constructions With Sets
7. Indexed and Fibred Sets
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13. Ends and Coends

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17. Abelian Categorical Hochschild Co/Homology

18. Categorical Hochschild Co/Homology

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19. Monoidal Categories

20. Monoidal Fibrations

21. Modules Over Monoidal Categories

22. Monoidal Limits and Colimits

23. Monoids in Monoidal Categories

24. Modules in Monoidal Categories

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- 26. Promonoidal Categories
- 27. 2-Groups
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- 45. Derived Categories

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-
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