

Indexed and Fibred Sets

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INTRODUCTION

This chapter contains a discussion of the un/straightening equivalence in the context of sets, as well as general constructions with indexed and fibred sets, like dependent sums and dependent products.

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1 Indexed Sets

1.1 Foundations

Let K be a set.

DEFINITION 1.1.1 ► INDEXED SETS

A **K -indexed set** is a functor $X: K_{\text{disc}} \rightarrow \text{Sets}$.

REMARK 1.1.2 ► UNWINDING DEFINITION 1.1.1

By **Categories**, **Proposition 1.5.1**, a **K -indexed set** consists of a K -indexed collection

$$X^\dagger: K \rightarrow \text{Obj}(\text{Sets}),$$

of sets, assigning a set $X_x \stackrel{\text{def}}{=} X_x$ to each element x of K .

DEFINITION 1.1.3 ► MORPHISMS OF INDEXED SETS

A **morphism of K -indexed sets from $X: K_{\text{disc}} \rightarrow \text{Sets}$ to $Y: K_{\text{disc}} \rightarrow \text{Sets}$** ¹ is a natural transformation

$$f: X \Rightarrow Y, \quad K_{\text{disc}} \begin{array}{c} \xrightarrow{X} \\ \Downarrow f \\ \xrightarrow{Y} \end{array} \text{Sets}$$

from X to Y .

¹*Further Terminology:* Also called a **K -indexed map of sets from X to Y** .

REMARK 1.1.4 ► UNWINDING DEFINITION 1.1.3

In detail, a **morphism of K -indexed sets** consists of a K -indexed collection

$$\{f_x: X_x \rightarrow Y_x\}_{x \in K}$$

of maps of sets.

DEFINITION 1.1.5 ► THE CATEGORY OF K -INDEXED SETS

The **category of K -indexed sets** is the category $\mathbf{ISets}(K)$ defined by

$$\mathbf{ISets}(K) \stackrel{\text{def}}{=} \text{Fun}(K_{\text{disc}}, \mathbf{Sets}).$$

REMARK 1.1.6 ► UNWINDING DEFINITION 1.1.5

In detail, the **category of K -indexed sets** is the category $\mathbf{ISets}(K)$ where

- *Objects.* The objects of $\mathbf{ISets}(K)$ are K -indexed sets;
- *Morphisms.* The morphisms of $\mathbf{ISets}(K)$ are morphisms of K -indexed sets;
- *Identities.* For each $X \in \text{Obj}(\mathbf{ISets}(K))$, the unit map

$$\mathbb{K}_X^{\mathbf{ISets}(K)} : \text{pt} \rightarrow \text{Hom}_{\mathbf{ISets}(K)}(X, X)$$

of $\mathbf{ISets}(K)$ at X is defined by

$$\text{id}_X^{\mathbf{ISets}(K)} \stackrel{\text{def}}{=} \{\text{id}_{X_x}\}_{x \in K};$$

- *Composition.* For each $X, Y, Z \in \text{Obj}(\mathbf{ISets}(K))$, the composition map

$$\circ_{X,Y,Z}^{\mathbf{ISets}(K)} : \text{Hom}_{\mathbf{ISets}(K)}(Y, Z) \times \text{Hom}_{\mathbf{ISets}(K)}(X, Y) \rightarrow \text{Hom}_{\mathbf{ISets}(K)}(X, Z)$$

of $\mathbf{ISets}(K)$ at (X, Y, Z) is defined by

$$\{g_x\}_{x \in K} \circ_{X,Y,Z}^{\mathbf{ISets}(K)} \{f_x\}_{x \in K} \stackrel{\text{def}}{=} \{g_x \circ f_x\}_{x \in K}.$$

DEFINITION 1.1.7 ► THE CATEGORY OF INDEXED SETS

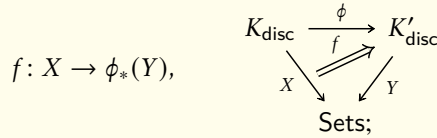
The **category of indexed sets** is the category \mathbf{ISets} defined as the Grothendieck construction of the functor $\mathbf{ISets} : \mathbf{Sets}^{\text{op}} \rightarrow \mathbf{Cats}$ of [Proposition 1.2.5](#):

$$\mathbf{ISets} \stackrel{\text{def}}{=} \int^{\mathbf{Sets}} \mathbf{ISets}.$$

REMARK 1.1.8 ► UNWINDING DEFINITION 1.1.7

In detail, the **category of indexed sets** is the category \mathbf{ISets} where

- *Objects.* The objects of \mathbf{ISets} are pairs (K, X) consisting of
 - *The Indexing Set.* A set K ;
 - *The Indexed Set.* A K -indexed set $X: K_{\text{disc}} \rightarrow \mathbf{Sets}$;
- *Morphisms.* A morphism of \mathbf{ISets} from (K, X) to (K', Y) is a pair (ϕ, f) consisting of
 - *The Reindexing Map.* A map of sets $\phi: K \rightarrow K'$;
 - *The Morphism of Indexed Sets.* A morphism of K -indexed sets $f: X \rightarrow \phi_*(Y)$ as in the diagram



- *Identities.* For each $(K, X) \in \text{Obj}(\mathbf{ISets})$, the unit map

$$\mathbb{1}_{(K,X)}^{\mathbf{ISets}}: \text{pt} \rightarrow \mathbf{ISets}((K, X), (K, X))$$

of \mathbf{ISets} at (K, X) is defined by

$$\text{id}_{(K,X)}^{\mathbf{ISets}} \stackrel{\text{def}}{=} (\text{id}_K, \text{id}_X).$$

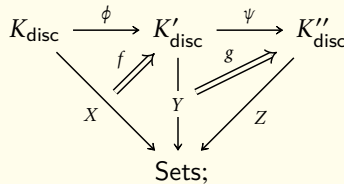
- *Composition.* For each $\mathbf{X} = (K, X), \mathbf{Y} = (K', Y), \mathbf{Z} = (K'', Z) \in \text{Obj}(\mathbf{ISets})$, the composition map

$$\circ_{\mathbf{X}, \mathbf{Y}, \mathbf{Z}}^{\mathbf{ISets}}: \mathbf{ISets}(\mathbf{Y}, \mathbf{Z}) \times \mathbf{ISets}(\mathbf{X}, \mathbf{Y}) \rightarrow \mathbf{ISets}(\mathbf{X}, \mathbf{Z})$$

of \mathbf{ISets} at $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ is defined by

$$(\psi, g) \circ (\phi, f) \stackrel{\text{def}}{=} (\psi \circ \phi, (g \star \text{id}_\phi) \circ f),$$

as in the diagram



for each $(\phi, f) \in \mathbf{ISets}(\mathbf{X}, \mathbf{Y})$ and each $(\psi, g) \in \mathbf{ISets}(\mathbf{Y}, \mathbf{Z})$.

1.2 Change of Indexing

Let $\phi: K \rightarrow K'$ be a function and let X be a K' -indexed set.

DEFINITION 1.2.1 ► CHANGE OF INDEXING OF INDEXED SETS

The **change of indexing of X to K** is the K -indexed set $\phi^*(X)$ defined by

$$\phi^*(X) \stackrel{\text{def}}{=} X \circ \phi_{\text{disc}}.$$

REMARK 1.2.2 ► UNWINDING DEFINITION 1.2.1

In detail, the **change of indexing of X to K** is the K -indexed set $\phi^*(X)$ defined by

$$\phi^*(X)_x \stackrel{\text{def}}{=} X_{\phi(x)}$$

for each $x \in K$.

PROPOSITION 1.2.3 ► FUNCTORIALITY OF CHANGE OF INDEXING

The assignment $X \mapsto \phi^*(X)$ defines a functor

$$\phi^*: \text{ISets}(K') \rightarrow \text{ISets}(K),$$

where

- *Action on Objects.* For each $X \in \text{Obj}(\text{ISets}(K'))$, we have

$$[\phi^*](X) \stackrel{\text{def}}{=} \phi^*(X);$$

- *Action on Morphisms.* For each $X, Y \in \text{Obj}(\text{ISets}(K'))$, the action on Hom-sets

$$\phi^*_{X,Y}: \text{Hom}_{\text{ISets}(K')}(X, Y) \rightarrow \text{Hom}_{\text{ISets}(K)}(\phi^*(X), \phi^*(Y))$$


of ϕ^* at (X, Y) is the map sending a morphism of K' -indexed sets

$$f = \{f_x: X_x \rightarrow Y_x\}_{x \in K'}$$

from X to Y to the morphism of K -indexed sets defined by

$$\phi^*(f) \stackrel{\text{def}}{=} \{f_{\phi(x)}: X_{\phi(x)} \rightarrow Y_{\phi(x)}\}_{x \in K}.$$

PROOF 1.2.4 ► PROOF OF PROPOSITION 1.2.3

Omitted. PROPOSITION 1.2.5 ► FUNCTORIALITY OF CATEGORIES OF K -INDEXED SETS

The assignment $K \mapsto \mathbf{ISets}(K)$ defines a functor

$$\mathbf{ISets}: \mathbf{Sets}^{\text{op}} \rightarrow \mathbf{Cats},$$

where

- *Action on Objects.* For each $K \in \text{Obj}(\mathbf{Sets})$, we have

$$[\mathbf{ISets}](K) \stackrel{\text{def}}{=} \mathbf{ISets}(K);$$

- *Action on Morphisms.* For each $K, K' \in \text{Obj}(\mathbf{Sets})$, the action on Hom-sets


$$\mathbf{ISets}_{K,K'}: \mathbf{Sets}^{\text{op}}(K, K') \rightarrow \text{Fun}(\mathbf{ISets}(K), \mathbf{ISets}(K'))$$

of \mathbf{ISets} at (K, K') is the map defined by

$$\mathbf{ISets}_{K,K'}(\phi) \stackrel{\text{def}}{=} \phi^*$$

for each $\phi \in \mathbf{Sets}^{\text{op}}(K, K')$.

PROOF 1.2.6 ► PROOF OF PROPOSITION 1.2.5

Omitted. 

1.3 Dependent Sums

Let $\phi: K \rightarrow K'$ be a function and let X be a K -indexed set.

DEFINITION 1.3.1 ► DEPENDENT SUMS OF INDEXED SETS

The **dependent sum of X** is the K' -indexed set $\Sigma_\phi(X)$ ¹ defined by

$$\Sigma_\phi(X) \stackrel{\text{def}}{=} \text{Lan}_\phi(X),$$

and hence given by

$$\Sigma_\phi(X)_x \cong \prod_{y \in \phi^{-1}(x)} X_y$$

for each $x \in K'$.

¹Further Notation: Also written $\phi_*(X)$.

PROPOSITION 1.3.2 ► FUNCTORIALITY OF DEPENDENT SUMS

The assignment $X \mapsto \Sigma_\phi(X)$ defines a functor

$$\Sigma_\phi : \mathbf{ISets}(K) \rightarrow \mathbf{ISets}(K'),$$

where

- *Action on Objects.* For each $X \in \mathbf{Obj}(\mathbf{ISets}(K))$, we have

$$[\Sigma_\phi](X) \stackrel{\text{def}}{=} \Sigma_\phi(X);$$

- *Action on Morphisms.* For each $X, Y \in \mathbf{Obj}(\mathbf{ISets}(K))$, the action on Hom-sets

$$\Sigma_{\phi|X,Y} : \mathbf{Hom}_{\mathbf{ISets}(K)}(X, Y) \rightarrow \mathbf{Hom}_{\mathbf{ISets}(K')}(\Sigma_\phi(X), \Sigma_\phi(Y))$$


of Σ_ϕ at (X, Y) is the map sending a morphism of K -indexed sets

$$f : X \rightarrow Y$$

to the morphism of K' -indexed sets defined by

$$\begin{aligned} \Sigma_\phi(f) &\stackrel{\text{def}}{=} \mathbf{Lan}_\phi(f); \\ &\cong \prod_{y \in \phi^{-1}(X)} f_y. \end{aligned}$$

PROOF 1.3.3 ► PROOF OF PROPOSITION 1.3.2

Omitted. 

1.4 Dependent Products

Let $\phi : K \rightarrow K'$ be a function and let X be a K -indexed set.

DEFINITION 1.4.1 ► DEPENDENT PRODUCTS OF INDEXED SETS

The **dependent product of X** is the K' -indexed set $\Pi_\phi(X)$ ¹ defined by

$$\Pi_\phi(X) \stackrel{\text{def}}{=} \text{Ran}_\phi(X),$$

and hence given by

$$\Pi_\phi(X)_x \cong \prod_{y \in \phi^{-1}(x)} X_y$$

for each $x \in K'$.

¹Further Notation: Also written $\phi_!(X)$.

PROPOSITION 1.4.2 ► FUNCTORIALITY OF DEPENDENT PRODUCTS

The assignment $X \mapsto \Pi_\phi(X)$ defines a functor

$$\Pi_\phi : \text{ISets}(K) \rightarrow \text{ISets}(K'),$$

where

- *Action on Objects.* For each $X \in \text{Obj}(\text{ISets}(K))$, we have

$$[\Pi_\phi](X) \stackrel{\text{def}}{=} \Pi_\phi(X);$$

- *Action on Morphisms.* For each $X, Y \in \text{Obj}(\text{ISets}(K))$, the action on Hom-sets

$$\Pi_{\phi|X,Y} : \text{Hom}_{\text{ISets}(K)}(X, Y) \rightarrow \text{Hom}_{\text{ISets}(K')}(\Pi_\phi(X), \Pi_\phi(Y))$$


of Π_ϕ at (X, Y) is the map sending a morphism of K -indexed sets

$$f : X \rightarrow Y$$

to the morphism of K' -indexed sets defined by

$$\begin{aligned} \Pi_\phi(f) &\stackrel{\text{def}}{=} \text{Ran}_\phi(f); \\ &\cong \prod_{y \in \phi^{-1}(x)} f_y. \end{aligned}$$

PROOF 1.4.3 ► PROOF OF PROPOSITION 1.4.2

Omitted. **1.5 Internal Homs**Let K be a set and let X and Y be K -indexed sets.

DEFINITION 1.5.1 ► INTERNAL HOM OF INDEXED SETS

The **internal Hom of indexed sets from X to Y** is the indexed set $\mathbf{Hom}_{\mathbf{Sets}(K)}(X, Y)$ defined by

$$\mathbf{Hom}_{\mathbf{Sets}(K)}(X, Y) \stackrel{\text{def}}{=} \mathbf{Sets}(X_x, Y_x)$$

for each $x \in K$.**1.6 Adjointness of Indexed Sets**Let $\phi: K \rightarrow K'$ be a map of sets.

PROPOSITION 1.6.1 ► ADJOINTNESS OF INDEXED SETS

We have a triple adjunction

$$(\Sigma_\phi \dashv \phi^* \dashv \Pi_\phi): \mathbf{ISets}(K) \leftarrow \phi^* \text{ --- } \mathbf{ISets}(K').$$

PROOF 1.6.2 ► PROOF OF PROPOSITION 1.6.1

This follows from [Kan Extensions, Item 2 of Proposition 1.1.6.](#) **2 Fibred Sets****2.1 Foundations**Let K be a set.

DEFINITION 2.1.1 ► FIBRED SETS

A **K -fibred set** is a pair (X, ϕ) consisting of¹

- *The Underlying Set.* A set X , called the **underlying set of** (X, ϕ) ;
- *The Fibration.* A map of sets $\phi: X \rightarrow K$.

¹*Further Terminology:* The **fib**re of (X, ϕ) **over** $x \in K$ is the set $\phi^{-1}(x)$ (also written ϕ_x) defined by

$$\phi^{-1}(x) \stackrel{\text{def}}{=} \text{pt} \times_{[x], K, \phi} X, \quad \begin{array}{ccc} \phi^{-1}(x) & \rightarrow & X \\ \downarrow & \lrcorner & \downarrow \phi \\ \text{pt} & \xrightarrow{[x]} & K. \end{array}$$

DEFINITION 2.1.2 ► MORPHISMS OF FIBRED SETS

A **morphism of K -fibred sets from (X, ϕ) to (Y, ψ)** is a function $f: X \rightarrow Y$ such that the diagram¹

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \phi \searrow & & \swarrow \psi \\ & K & \end{array}$$

commutes.

¹*Further Terminology:* The **transport map associated to f at $x \in K$** is the function

$$f_x^*: \phi^{-1}(x) \rightarrow \psi^{-1}(x)$$

given by the dashed map in the diagram

$$\begin{array}{ccccc} \phi^{-1}(x) & \xrightarrow{\quad} & X & \xrightarrow{f} & Y \\ \downarrow & \dashrightarrow & \downarrow \phi & & \downarrow \psi \\ \psi^{-1}(x) & \xrightarrow{\quad} & Y & & \\ \downarrow & \dashrightarrow & \downarrow & & \\ \text{pt} & \xrightarrow{[x]} & K & & \\ \parallel & & \parallel & & \\ \text{pt} & \xrightarrow{[x]} & K. & & \end{array}$$

DEFINITION 2.1.3 ► THE CATEGORY OF K -FIBRED SETS

The **category of K -fibred sets** is the category $\text{FibSets}(K)$ defined as the slice category $\text{Sets}/_K$ of Sets over K :

$$\text{FibSets}(K) \stackrel{\text{def}}{=} \text{Sets}/_K.$$

REMARK 2.1.4 ► UNWINDING DEFINITION 2.1.3

In detail $\text{FibSets}(K)$ is the category where

- *Objects.* The objects of $\text{FibSets}(K)$ are pairs (X, ϕ) consisting of
 - *The Fibred Set.* A set X ;
 - *The Fibration.* A function $\phi: X \rightarrow K$;
- *Morphisms.* A morphism of $\text{FibSets}(K)$ from (X, ϕ) to (Y, ψ) is a function $f: X \rightarrow Y$ making the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \phi \searrow & & \swarrow \psi \\ & X & \end{array}$$

commute;

- *Identities.* For each $(X, \phi) \in \text{Obj}(\text{FibSets}(K))$, the unit map

$$\text{id}_{(X, \phi)}^{\text{FibSets}(K)}: \text{pt} \rightarrow \text{Hom}_{\text{FibSets}(K)}((X, \phi), (X, \phi))$$

of $\text{FibSets}(K)$ at (X, ϕ) is given by

$$\text{id}_{(X, \phi)}^{\text{FibSets}(K)} \stackrel{\text{def}}{=} \text{id}_X,$$

as witnessed by the commutativity of the diagram

$$\begin{array}{ccc} X & \xrightarrow{\text{id}_X} & X \\ \phi \searrow & & \swarrow \phi \\ & K & \end{array}$$

in Sets;

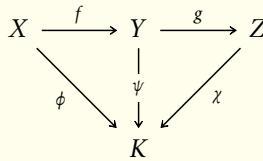
- *Composition.* For each $\mathbf{X} = (X, \phi), \mathbf{Y} = (Y, \psi), \mathbf{Z} = (Z, \chi) \in \text{Obj}(\text{FibSets}(K))$, the composition map

$$\circ_{\mathbf{X}, \mathbf{Y}, \mathbf{Z}}^{\text{FibSets}(K)} : \text{Hom}_{\text{FibSets}(K)}(\mathbf{Y}, \mathbf{Z}) \times \text{Hom}_{\text{FibSets}(K)}(\mathbf{X}, \mathbf{Y}) \rightarrow \text{Hom}_{\text{FibSets}(K)}(\mathbf{X}, \mathbf{Z})$$

of $\text{FibSets}(K)$ at $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ is defined by

$$\circ_{\mathbf{X}, \mathbf{Y}, \mathbf{Z}}^{\text{FibSets}(K)} \stackrel{\text{def}}{=} \circ_{X, Y, Z}^{\text{Sets}}$$

as witnessed by the commutativity of the diagram



in Sets .

DEFINITION 2.1.5 ► THE CATEGORY OF FIBRED SETS

The **category of fibred sets** is the category FibSets defined as the Grothendieck construction of the functor $\text{FibSets} : \text{Sets}^{\text{op}} \rightarrow \text{Cats}$ of [Proposition 2.2.4](#):

$$\text{FibSets} \stackrel{\text{def}}{=} \int^{\text{Sets}} \text{FibSets}.$$

REMARK 2.1.6 ► UNWINDING DEFINITION 2.1.5

In detail, the **category of fibred sets** is the category FibSets where

- *Objects.* The objects of FibSets are pairs $(K, (X, \phi_X))$ consisting of
 - *The Base Set.* A set K ;
 - *The Fibred Set.* A K -fibred set $\phi_X : X \rightarrow K$;
- *Morphisms.* A morphism of FibSets from $(K, (X, \phi_X))$ to $(K', (Y, \phi_Y))$ is a pair (ϕ, f) consisting of
 - *The Base Map.* A map of sets $\phi : K \rightarrow K'$;

· *The Morphism of Fibred Sets.* A morphism of K -fibred sets

$$f: (X, \phi_X) \rightarrow \phi_Y^*(Y), \quad \begin{array}{ccc} X & \xrightarrow{f} & Y \times_{K'} K \\ & \searrow \phi_X & \swarrow \text{pr}_2 \\ & & K; \end{array}$$

· *Identities.* For each $(K, X) \in \text{Obj}(\text{FibSets})$, the unit map

$$\mathbb{1}_{(K,X)}^{\text{FibSets}}: \text{pt} \rightarrow \text{FibSets}((K, X), (K, X))$$

of FibSets at (K, X) is defined by

$$\text{id}_{(K,X)}^{\text{FibSets}} \stackrel{\text{def}}{=} (\text{id}_K, \sim),$$

where \sim is the isomorphism $X \rightarrow X \times_K K$ as in the diagram

$$\begin{array}{ccc} X & \xrightarrow{\sim} & X \times_K K \\ & \searrow \phi_X & \swarrow \text{pr}_2 \\ & & K; \end{array}$$

· *Composition.* For each $\mathbf{X} = (K, X), \mathbf{Y} = (K', Y), \mathbf{Z} = (K'', Z) \in \text{Obj}(\text{FibSets})$, the composition map

$$\circ_{\mathbf{X}, \mathbf{Y}, \mathbf{Z}}^{\text{FibSets}}: \text{FibSets}(\mathbf{Y}, \mathbf{Z}) \times \text{FibSets}(\mathbf{X}, \mathbf{Y}) \rightarrow \text{FibSets}(\mathbf{X}, \mathbf{Z})$$

of FibSets at $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ is defined by

$$g \circ_{\mathbf{X}, \mathbf{Y}, \mathbf{Z}}^{\text{FibSets}} f \stackrel{\text{def}}{=} (g \times_{K'} \text{id}_K) \circ f$$

as in the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \times_{K'} K \xrightarrow{g \times_{K'} \text{id}_K} \overbrace{(Z \times_{K''} K')}^{\cong Z \times_{K'} K} \\ & \searrow \phi_X & \downarrow \text{pr}_2 \swarrow \text{pr}_2 \\ & & K; \end{array}$$

for each $f \in \text{FibSets}(\mathbf{X}, \mathbf{Y})$ and each $g \in \text{FibSets}(\mathbf{Y}, \mathbf{Z})$.

2.2 Change of Base

Let $f: K \rightarrow K'$ be a function and let (X, ϕ) be a K' -fibred set.

DEFINITION 2.2.1 ► CHANGE OF BASE FOR FIBRED SETS

The **change of base of (X, ϕ) to K** is the K -fibred set $f^*(X)$ defined by

$$f^*(X) \stackrel{\text{def}}{=} (K \times_{K'} X, \text{pr}_1),$$

$$\begin{array}{ccc} f^*(X) & \xrightarrow{\text{pr}_2} & X \\ \text{pr}_1 \downarrow & \lrcorner & \downarrow \phi \\ K & \xrightarrow{f} & K' \end{array}$$

PROPOSITION 2.2.2 ► FUNCTORIALITY OF CHANGE OF BASE

The assignment $X \mapsto f^*(X)$ defines a functor

$$f^*: \text{FibSets}(K') \rightarrow \text{FibSets}(K),$$

where

- *Action on Objects.* For each $(X, \phi) \in \text{Obj}(\text{FibSets}(K'))$, we have

$$f^*(X, \phi) \stackrel{\text{def}}{=} f^*(X);$$

- *Action on Morphisms.* For each $(X, \phi), (Y, \psi) \in \text{Obj}(\text{FibSets}(K'))$, the action on Hom-sets

$$f^*_{X,Y}: \text{Hom}_{\text{FibSets}(K')} (X, Y) \rightarrow \text{Hom}_{\text{FibSets}(K)} (f^*(X), f^*(Y))$$


of f^* at $((X, \phi), (Y, \psi))$ is the map sending a morphism of K' -fibred sets

$$g: (X, \phi) \rightarrow (Y, \psi)$$

to the morphism of K -fibred sets given by the dashed morphism in the diagram

$$\begin{array}{ccccc} f^*(X) & \longrightarrow & X & & \\ \downarrow & \lrcorner & \downarrow \phi & \searrow g & \\ & & f^*(Y) & \xrightarrow{\quad} & Y \\ & & \downarrow & & \downarrow \psi \\ K & \xrightarrow{f} & K' & & \\ \parallel & & \parallel & & \\ K & \xrightarrow{f} & K' & & \end{array}$$

PROOF 2.2.3 ► PROOF OF PROPOSITION 2.2.2

Omitted. PROPOSITION 2.2.4 ► FUNCTORIALITY OF CATEGORIES OF K -FIBRED SETS

The assignment $K \mapsto \text{FibSets}(K)$ defines a functor

$$\text{FibSets}: \text{Sets}^{\text{op}} \rightarrow \text{Cats},$$

where

- *Action on Objects.* For each $K \in \text{Obj}(\text{Sets})$, we have

$$[\text{FibSets}](K) \stackrel{\text{def}}{=} \text{FibSets}(K);$$

- *Action on Morphisms.* For each $K, K' \in \text{Obj}(\text{Sets})$, the action on Hom-sets

$$\text{Sets}_{/(-)|K,K'}: \text{Sets}^{\text{op}}(K, K') \rightarrow \text{Fun}(\text{FibSets}(K), \text{FibSets}(K'))$$


of $\text{Sets}_{/(-)}$ at (K, K') is the map sending a map of sets $f: K \rightarrow K'$ to the functor

$$\text{Sets}_{/f}: \text{FibSets}(K') \rightarrow \text{FibSets}(K)$$

defined by

$$\text{Sets}_{/f} \stackrel{\text{def}}{=} f^*.$$

PROOF 2.2.5 ► PROOF OF PROPOSITION 2.2.4

Omitted. 

2.3 Dependent Sums

Let $f: K \rightarrow K'$ be a function and let (X, ϕ) be a K -fibred set.

DEFINITION 2.3.1 ► DEPENDENT SUMS FOR FIBRED SETS

The **dependent sum**¹ of (X, ϕ) is the K' -fibred set $\Sigma_f(X)$ ² defined by

$$\begin{aligned} \Sigma_f(X) &\stackrel{\text{def}}{=} (\Sigma_f(X), \Sigma_f(\phi)) \\ &\stackrel{\text{def}}{=} (X, f \circ \phi). \end{aligned}$$

¹The name “dependent sum” comes from the fact that the fibre $\Sigma_f(\phi)^{-1}(x)$ of $\Sigma_f(X)$ at $x \in K'$ is given by

$$\Sigma_f(\phi)^{-1}(x) \cong \coprod_{y \in f^{-1}(x)} \phi^{-1}(y);$$

see [Item 2](#) of [Proposition 2.3.2](#).

²*Further Notation:* Also written $f_*(X)$.

PROPOSITION 2.3.2 ► PROPERTIES OF DEPENDENT SUMS OF FIBRED SETS

Let $f: K \rightarrow K'$ be a function.

1. *Functoriality.* The assignment $X \mapsto \Sigma_f(X)$ defines a functor

$$\Sigma_f: \text{FibSets}(K) \rightarrow \text{FibSets}(K'),$$

where

- *Action on Objects.* For each $(X, \phi) \in \text{Obj}(\text{FibSets}(K))$, we have

$$\Sigma_f(X, \phi) \stackrel{\text{def}}{=} (\Sigma_f(X), \Sigma_f(\phi));$$

- *Action on Morphisms.* For each $(X, \phi), (Y, \psi) \in \text{Obj}(\text{FibSets}(K))$, the action on Hom-sets

$$\Sigma_f|_{X,Y}: \text{Hom}_{\text{FibSets}(K)}(X, Y) \rightarrow \text{Hom}_{\text{FibSets}(K')}(\Sigma_f(X), \Sigma_f(Y))$$

of Σ_f at $((X, \phi), (Y, \psi))$ is the map sending a morphism of K -fibred sets

$$g: (X, \phi) \rightarrow (Y, \psi)$$

to the morphism of K' -fibred sets defined by

$$\Sigma_f(g) \stackrel{\text{def}}{=} g.$$

2. *Interaction With Fibres.* We have a bijection of sets

$$\Sigma_f(\phi)^{-1}(x) \cong \coprod_{y \in f^{-1}(x)} \phi^{-1}(y)$$

for each $x \in K'$.

PROOF 2.3.3 ► PROOF OF PROPOSITION 2.3.2

Item 1: Functoriality

Omitted.

Item 2: Interaction With Fibres

Indeed, we have

$$\begin{aligned}
 \Sigma_f(\phi)^{-1}(x) &\stackrel{\text{def}}{=} \text{pt} \times_{[x], K', f \circ \phi} X \\
 &\cong \{(a, y) \in X \times K \mid f(\phi(a)) = x\} \\
 &\cong \coprod_{y \in f^{-1}(x)} \phi^{-1}(y)
 \end{aligned}$$

for each $x \in K'$.

2.4 Dependent Products

Let $f: K \rightarrow K'$ be a function and let (X, ϕ) be a K -fibred set.

DEFINITION 2.4.1 ► DEPENDENT PRODUCTS FOR FIBRED SETS

The **dependent product**¹ of (X, ϕ) is the K' -fibred set $\Pi_f(X)$ ² consisting of³

- *The Underlying Set.* The set $\Pi_f(X)$ defined by

$$\begin{aligned}
 \Pi_f(X) &\stackrel{\text{def}}{=} \coprod_{x \in K'} \Gamma_{f^{-1}(x)}^\phi \left(\phi^{-1}(f^{-1}(x)) \right) \\
 &\stackrel{\text{def}}{=} \left\{ (x, h) \in \coprod_{x \in K'} \text{Sets}(f^{-1}(x), \phi^{-1}(f^{-1}(x))) \mid \phi \circ h = \text{id}_{f^{-1}(x)} \right\};
 \end{aligned}$$

- *The Fibration.* The map of sets

$$\Pi_f(\phi): \coprod_{x \in K'} \Gamma_{f^{-1}(x)}^\phi \left(\phi^{-1}(f^{-1}(x)) \right) \rightarrow K$$

defined by sending a map $h: f^{-1}(x) \rightarrow \phi^{-1}(f^{-1}(x))$ to its index $x \in K$.

¹The name “dependent product” comes from the fact that the fibre $\Pi_f(\phi)^{-1}(x)$ of $\Pi_f(X)$ at $x \in K'$ is given by

$$\Pi_f(\phi)^{-1}(x) \cong \prod_{y \in f^{-1}(x)} \phi^{-1}(y);$$

see [Item 2 of Proposition 2.4.3](#).

²Further Notation: Also written $f_!(X)$.

³We can also define dependent products via the internal **Hom** in $\mathbf{FibSets}(K')$:

$$\Pi_f(X, \phi) \stackrel{\text{def}}{=} \left(K' \times_{\mathbf{Hom}_{\mathbf{FibSets}(K')}(f, f)} \mathbf{Hom}_{\mathbf{Sets}/K'}(f, f \circ \phi), \text{pr}_1 \right),$$

$$\begin{array}{ccc} \Pi_f(X, \phi) & \xrightarrow{\text{pr}_2} & \mathbf{Hom}_{\mathbf{Sets}/K'}(f, f \circ \phi) \\ \text{pr}_1 \downarrow & \lrcorner & \downarrow \\ K' & \xrightarrow{I} & \mathbf{Hom}_{\mathbf{FibSets}(K')}(f, f), \end{array}$$

where the bottom map is defined by

$$I(x) \stackrel{\text{def}}{=} \text{id}_{f^{-1}(x)}$$

for each $x \in K'$.

EXAMPLE 2.4.2 ► EXAMPLES OF DEPENDENT PRODUCTS OF SETS

Here are some examples of dependent products of sets.

1. *Spaces of Sections.* Let $K = X$, $K' = \text{pt}$, and let $\phi: E \rightarrow X$ be a map of sets. We have a bijection of sets

$$\begin{aligned} \Pi_{1_X}(\phi) &\cong \Gamma_X(\phi) \\ &\cong \{h \in \mathbf{Sets}(X, E) \mid \phi \circ h = \text{id}_X\}. \end{aligned}$$

2. *Function Spaces.* Let $K = K' = \text{pt}$. We have a bijection of sets

$$\mathbf{Sets}(X, Y) \cong \Pi_{1_X}(!_X^*(Y)).$$

PROPOSITION 2.4.3 ► PROPERTIES OF DEPENDENT PRODUCTS OF FIBRED SETS

Let $f: K \rightarrow K'$ be a function.

1. *Functoriality.* The assignment $X \mapsto \Pi_f(X)$ defines a functor

$$\Pi_f: \mathbf{FibSets}(K) \rightarrow \mathbf{FibSets}(K'),$$

where

- *Action on Objects.* For each $(X, \phi) \in \text{Obj}(\mathbf{FibSets}(K))$, we have

$$\Pi_f(X, \phi) \stackrel{\text{def}}{=} \Pi_f(X);$$

- *Action on Morphisms.* For each $(X, \phi), (Y, \psi) \in \text{Obj}(\mathbf{FibSets}(K))$, the action on Hom-sets

$$\Pi_{f|X, Y}: \mathbf{Hom}_{\mathbf{FibSets}(K)}(X, Y) \rightarrow \mathbf{Hom}_{\mathbf{FibSets}(K')}(\Pi_f(X), \Pi_f(Y))$$

of Π_f at $((X, \phi), (Y, \psi))$ is the map sending a morphism of K -fibred sets

$$g: (X, \phi) \rightarrow (Y, \psi)$$

to the morphism of K' -fibred sets from

$$\Pi_f(X) \stackrel{\text{def}}{=} \left\{ (x, h) \in \prod_{x \in K'} \text{Sets}(f^{-1}(x), \phi^{-1}(f^{-1}(x))) \mid \phi \circ h = \text{id}_{f^{-1}(x)} \right\};$$

to

$$\Pi_f(Y) \stackrel{\text{def}}{=} \left\{ (x, h) \in \prod_{x \in K'} \text{Sets}(f^{-1}(x), \psi^{-1}(f^{-1}(x))) \mid \psi \circ h = \text{id}_{f^{-1}(x)} \right\};$$

induced by the composition

$$\begin{aligned} \text{Sets}(f^{-1}(x), \phi^{-1}(f^{-1}(x))) &= \text{Sets}(f^{-1}(x), [\psi \circ g]^{-1}(f^{-1}(x))) \\ &= \text{Sets}(f^{-1}(x), g^{-1}(\psi^{-1}(f^{-1}(x)))) \\ &\xrightarrow{g^*} \text{Sets}(f^{-1}(x), g(g^{-1}(\psi^{-1}(f^{-1}(x))))) \\ &\xrightarrow{\iota^*} \text{Sets}(f^{-1}(x), \psi^{-1}(f^{-1}(x))), \end{aligned}$$

where $\iota: g(g^{-1}(\psi^{-1}(f^{-1}(x)))) \hookrightarrow \psi^{-1}(f^{-1}(x))$ is the canonical inclusion.¹

2. *Interaction With Fibres.* We have a bijection of sets

$$\Pi_f(\phi)^{-1}(x) \cong \prod_{y \in f^{-1}(x)} \phi^{-1}(y)$$

for each $x \in K'$.

¹Note that the section condition is satisfied: given $(x, h) \in \Pi_f(X)$, we have

$$\begin{aligned} \psi \circ [\Pi_f(g)](h) &\stackrel{\text{def}}{=} \psi \circ (g \circ h) \\ &= (\psi \circ g) \circ h \\ &= \phi \circ h \\ &= \text{id}_{f^{-1}(x)}. \end{aligned}$$

PROOF 2.4.4 ► PROOF OF PROPOSITION 2.4.3


Item 1: Functoriality

Omitted.

Item 2: Interaction With Fibres

Indeed, we have

$$\begin{aligned}
\Pi_f(\phi)^{-1}(x) &\stackrel{\text{def}}{=} \{(y, h) \in \Pi_f(X) \mid [\Pi_f(\phi)](h) = x\} \\
&\stackrel{\text{def}}{=} \{(y, h) \in \Pi_f(X) \mid y = x\} \\
&\cong \left\{ h \in \text{Sets}\left(f^{-1}(x), \phi^{-1}\left(f^{-1}(x)\right)\right) \mid \phi \circ h = \text{id}_{f^{-1}(x)} \right\} \\
&\cong \prod_{y \in f^{-1}(x)} \phi^{-1}(y)
\end{aligned}$$

for each $x \in K'$. 

2.5 Internal Homs

Let K be a set and let (X, ϕ) and (Y, ψ) be K -fibred sets.

DEFINITION 2.5.1 ► INTERNAL HOM OF FIBRED SETS

The **internal Hom of fibred sets from (X, ϕ) to (Y, ψ)** is the fibred set $\mathbf{Hom}_{\text{FibSets}(K)}(X, Y)$ consisting of

- *The Underlying Set.* The set $\mathbf{Hom}_{\text{FibSets}(K)}(X, Y)$ defined by

$$\mathbf{Hom}_{\text{FibSets}(K)}(X, Y) \stackrel{\text{def}}{=} \prod_{x \in K} \text{Sets}\left(\phi^{-1}(x), \psi^{-1}(x)\right);$$

- *The Fibration.* The map of sets¹

$$\begin{aligned}
\phi \mathbf{Hom}_{\text{FibSets}(K)}(X, Y) : \underbrace{\mathbf{Hom}_{\text{FibSets}(K)}(X, Y)}_{\prod_{x \in K} \text{Sets}\left(\phi^{-1}(x), \psi^{-1}(x)\right)} &\rightarrow K
\end{aligned}$$

defined by sending a map $f: \phi^{-1}(x) \rightarrow \psi^{-1}(x)$ to its index $x \in K$.

¹The fibres of the internal Hom of $\text{FibSets}(K)$ are precisely the sets $\text{Sets}\left(\phi^{-1}(x), \psi^{-1}(x)\right)$, i.e. we have

$$\phi \mathbf{Hom}_{\text{FibSets}(K)}(X, Y)|_x \cong \text{Sets}\left(\phi^{-1}(x), \psi^{-1}(x)\right)$$

for each $x \in K$.

2.6 Adjointness for Fibred Sets

Let $f: K \rightarrow K'$ be a map of sets.


PROPOSITION 2.6.1 ► ADJOINTNESS FOR FIBRED SETS

We have a triple adjunction

$$(\Sigma_f \dashv f^* \dashv \Pi_f): \text{FibSets}(K) \leftarrow \text{FibSets}(K')$$

$\begin{array}{ccc} & \Sigma_f & \\ \uparrow \perp & \curvearrowright & \\ & f^* & \\ \downarrow \perp & \curvearrowleft & \\ & \Pi_f & \end{array}$

PROOF 2.6.2 ► PROOF OF PROPOSITION 2.6.1

Omitted. 

3 Un/Straightening for Indexed and Fibred Sets

3.1 Straightening for Fibred Sets

Let K be a set and let (X, ϕ) be a K -fibred set.

DEFINITION 3.1.1 ► THE STRAIGHTENING OF A FIBRED SET

The **straightening of** (X, ϕ) is the K -indexed set

$$\text{St}_K(X, \phi): K_{\text{disc}} \rightarrow \text{Sets}$$

defined by

$$\text{St}_K(X, \phi)_x \stackrel{\text{def}}{=} \phi^{-1}(x)$$

for each $x \in K$.

PROPOSITION 3.1.2 ► PROPERTIES OF STRAIGHTENING FOR FIBRED SETS

Let K be a set.

1. *Functoriality.* The assignment $(X, \phi) \mapsto \text{St}_K(X, \phi)$ defines a functor

$$\text{St}_K: \text{FibSets}(K) \rightarrow \text{ISets}(K)$$

- *Action on Objects.* For each $(X, \phi) \in \text{Obj}(\text{FibSets}(K))$, we have

$$[\text{St}_K](X, \phi) \stackrel{\text{def}}{=} \text{St}_K(X, \phi);$$

- *Action on Morphisms.* For each $(X, \phi), (Y, \psi) \in \text{Obj}(\text{FibSets}(K))$, the action on Hom-sets

$$\text{St}_{K|X,Y}: \text{Hom}_{\text{FibSets}(K)}(X, Y) \rightarrow \text{Hom}_{\text{ISets}(K)}(\text{St}_K(X), \text{St}_K(Y))$$

of St_K at (X, Y) is given by sending a morphism

$$f: (X, \phi) \rightarrow (Y, \psi)$$

of K -fibred sets to the morphism

$$\text{St}_K(f): \text{St}_K(X, \phi) \rightarrow \text{St}_K(Y, \psi)$$

of K -indexed sets defined by

$$\text{St}_K(f) \stackrel{\text{def}}{=} \{f_x^*\}_{x \in K},$$

where f_x^* is the transport map associated to f at $x \in K$ of **Definition 2.1.2**.

2. *Interaction With Change of Base/Indexing.* Let $f: K \rightarrow K'$ be a map of sets. The diagram

$$\begin{array}{ccc} \text{FibSets}(K') & \xrightarrow{f^*} & \text{FibSets}(K) \\ \text{St}_{K'} \downarrow & & \downarrow \text{St}_K \\ \text{ISets}(K') & \xrightarrow{f^*} & \text{ISets}(K) \end{array}$$

commutes.

3. *Interaction With Dependent Sums.* Let $f: K \rightarrow K'$ be a map of sets. The diagram

$$\begin{array}{ccc} \text{FibSets}(K) & \xrightarrow{\Sigma_f} & \text{FibSets}(K') \\ \text{St}_K \downarrow & & \downarrow \text{St}_{K'} \\ \text{ISets}(K) & \xrightarrow{\Sigma_f} & \text{ISets}(K') \end{array}$$

commutes.

4. *Interaction With Dependent Products.* Let $f : K \rightarrow K'$ be a map of sets. The diagram

$$\begin{array}{ccc} \mathbf{Sets}/_K & \xrightarrow{\Pi_f} & \mathbf{FibSets}(K') \\ \mathbf{St}_K \downarrow & & \downarrow \mathbf{St}_{K'} \\ \mathbf{ISets}(K) & \xrightarrow{\Pi_f} & \mathbf{ISets}(K') \end{array}$$

commutes.

PROOF 3.1.3 ► PROOF OF PROPOSITION 3.1.2

Item 1: Functoriality

Omitted.

Item 2: Interaction With Change of Base/Indexing

Indeed, we have

$$\begin{aligned} \mathbf{St}_K(f^*(X, \phi))_x &\stackrel{\text{def}}{=} \mathbf{St}_K(K \times_{K'} X)_x \\ &\stackrel{\text{def}}{=} \left(\text{pr}_1^{K \times_{K'} X} \right)^{-1}(x) \\ &= \left\{ (k, y) \in K \times_{K'} X \mid \text{pr}_1^{K \times_{K'} X}(k, y) = x \right\} \\ &= \{ (k, y) \in K \times_{K'} X \mid k = x \} \\ &= \{ (k, y) \in K \times X \mid k = x \text{ and } f(k) = \phi(y) \} \\ &\cong \{ y \in X \mid \phi(y) = f(x) \} \\ &= \phi^{-1}(f(x)) \\ &\stackrel{\text{def}}{=} f^*(\phi^{-1}(x)) \\ &\stackrel{\text{def}}{=} f^*(\mathbf{St}_{K'}(X, \phi)_x) \end{aligned}$$

for each $(X, \phi) \in \text{Obj}(\mathbf{FibSets}(K'))$ and each $x \in K$, and similarly for morphisms.

Item 3: Interaction With Dependent Sums

Indeed, we have

$$\mathbf{St}_{K'}(\Sigma_f(X, \phi))_x \stackrel{\text{def}}{=} \Sigma_f(\phi)^{-1}(x)$$


$$\begin{aligned}
&\cong \coprod_{\substack{y \in X \\ f(y)=x}} \phi^{-1}(y) \\
&\cong \Sigma_f(\phi^{-1}(x)) \\
&\stackrel{\text{def}}{=} \Sigma_f(\text{St}_K(X, \phi)_x)
\end{aligned}$$

for each $(X, \phi) \in \text{Obj}(\text{FibSets}(K))$ and each $x \in K'$, where we have used **Item 2** of **Proposition 2.3.2** for the first bijection, and similarly for morphisms.

Item 4: Interaction With Dependent Products

Indeed, we have

$$\begin{aligned}
\text{St}_{K'}(\Pi_f(X, \phi))_x &\stackrel{\text{def}}{=} \Pi_f(\phi)^{-1}(x) \\
&\cong \prod_{\substack{y \in X \\ f(y)=x}} \phi^{-1}(y) \\
&\cong \Pi_f(\phi^{-1}(x)) \\
&\stackrel{\text{def}}{=} \Pi_f(\text{St}_K(X, \phi)_x)
\end{aligned}$$

for each $(X, \phi) \in \text{Obj}(\text{FibSets}(K))$ and each $x \in K'$, where we have used **Item 2** of **Proposition 2.4.3** for the first bijection, and similarly for morphisms. 

3.2 Unstraightening for Indexed Sets

Let K be a set and let X be a K -indexed set.

DEFINITION 3.2.1 ► THE UNSTRAIGHTENING OF AN INDEXED SET

The **unstraightening of X** is the K -fibred set

$$\phi_{\text{Un}_K}: \text{Un}_K(X) \rightarrow K$$

consisting of

- *The Underlying Set.* The set $\text{Un}_K(X)$ defined by

$$\text{Un}_K(X) \stackrel{\text{def}}{=} \prod_{x \in K} X_x;$$

· *The Fibration.* The map of sets

$$\phi_{\text{Un}_K}: \text{Un}_K(X) \rightarrow K$$

defined by sending an element of $\coprod_{x \in K} X_x$ to its index in K .

PROPOSITION 3.2.2 ► PROPERTIES OF UNSTRAIGHTENING FOR INDEXED SETS

Let K be a set.

1. *Functoriality.* The assignment $X \mapsto \text{Un}_K(X)$ defines a functor

$$\text{Un}_K: \text{ISets}(K) \rightarrow \text{FibSets}(K)$$

· *Action on Objects.* For each $X \in \text{Obj}(\text{ISets}(K))$, we have

$$[\text{Un}_K](X) \stackrel{\text{def}}{=} \text{Un}_K(X);$$

· *Action on Morphisms.* For each $X, Y \in \text{Obj}(\text{ISets}(K))$, the action on Hom-sets

$$\text{Un}_{K|X,Y}: \text{Hom}_{\text{ISets}(K)}(X, Y) \rightarrow \text{Hom}_{\text{FibSets}(K)}(\text{Un}_K(X), \text{Un}_K(Y))$$

of Un_K at (X, Y) is defined by

$$\text{Un}_{K|X,Y}(f) \stackrel{\text{def}}{=} \coprod_{x \in K} f_x^*.$$

2. *Interaction With Fibres.* We have a bijection of sets

$$\phi_{\text{Un}_K}^{-1}(x) \cong X_x$$

for each $x \in K$.

3. *As a Pullback.* We have a bijection of sets

$$\begin{array}{ccc} \text{Un}_K(X) & \rightarrow & \text{Sets}_* \\ \cong K_{\text{disc}} \times_{\text{Sets}} \text{Sets}_* & \lrcorner & \downarrow \text{忘} \\ & & K_{\text{disc}} \xrightarrow{X} \text{Sets}. \end{array}$$

4. *As a Colimit.* We have a bijection of sets

$$\text{Un}_K(X) \cong \text{colim}(X).$$

5. *Interaction With Change of Indexing/Base.* Let $f : K \rightarrow K'$ be a map of sets. The diagram

$$\begin{array}{ccc} \mathbf{ISets}(K') & \xrightarrow{f^*} & \mathbf{ISets}(K) \\ \mathbf{Un}_{K'} \downarrow & & \downarrow \mathbf{Un}_K \\ \mathbf{FibSets}(K') & \xrightarrow{f^*} & \mathbf{FibSets}(K) \end{array}$$

commutes.

6. *Interaction With Dependent Sums.* Let $f : K \rightarrow K'$ be a map of sets. The diagram

$$\begin{array}{ccc} \mathbf{ISets}(K) & \xrightarrow{\Sigma_f} & \mathbf{ISets}(K') \\ \mathbf{Un}_K \downarrow & & \downarrow \mathbf{Un}_{K'} \\ \mathbf{FibSets}(K) & \xrightarrow{\Sigma_f} & \mathbf{FibSets}(K') \end{array}$$

commutes.

7. *Interaction With Dependent Products.* Let $f : K \rightarrow K'$ be a map of sets. The diagram

$$\begin{array}{ccc} \mathbf{ISets}(K) & \xrightarrow{\Pi_f} & \mathbf{ISets}(K') \\ \mathbf{Un}_K \downarrow & & \downarrow \mathbf{Un}_{K'} \\ \mathbf{FibSets}(K) & \xrightarrow{\Pi_f} & \mathbf{FibSets}(K') \end{array}$$

commutes.

PROOF 3.2.3 ► PROOF OF PROPOSITION 3.2.2

Item 1: Functoriality

Omitted.

Item 2: Interaction With Fibres

Omitted.

Item 3: As a Pullback

Omitted.

Item 4: As a Colimit

Clear.

Item 5: Interaction With Change of Indexing/Base

Indeed, we have

$$\begin{aligned}
 \mathrm{Un}_K(f^*(X)) &\stackrel{\mathrm{def}}{=} \mathrm{Un}_K(X \circ f) \\
 &\stackrel{\mathrm{def}}{=} \coprod_{x \in K} X_{f(x)} \\
 &\cong \left\{ (x, (y, a)) \in K \times \prod_{y \in K'} X_y \mid f(x) = y \right\} \\
 &\cong K \times_{K'} \prod_{y \in K'} X_y \\
 &\stackrel{\mathrm{def}}{=} K \times_{K'} \mathrm{Un}_{K'}(X) \\
 &\stackrel{\mathrm{def}}{=} f^*(\mathrm{Un}_{K'}(X))
 \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathrm{ISets}(K'))$. Similarly, it can be shown that we also have $\mathrm{Un}_K(f^*(\phi)) = f^*(\mathrm{Un}_{K'}(\phi))$ and that $\mathrm{Un}_K \circ f^* = f^* \circ \mathrm{Un}_{K'}$ also holds on morphisms.

Item 6: Interaction With Dependent Sums

Indeed, we have

$$\begin{aligned}
 \mathrm{Un}_{K'}(\Sigma_f(X)) &\stackrel{\mathrm{def}}{=} \prod_{x \in K'} \Sigma_f(X)_x \\
 &\cong \prod_{x \in K'} \prod_{y \in f^{-1}(x)} X_y \\
 &\cong \prod_{y \in K} X_y \\
 &\cong \mathrm{Un}_K(X) \\
 &\stackrel{\mathrm{def}}{=} \Sigma_f(\mathrm{Un}_K(X))
 \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathrm{ISets}(K))$, where we have used [Item 2](#) of [Proposition 2.3.2](#) for the first bijection. Similarly, it can be shown that we also have $\mathrm{Un}_{K'}(\Sigma_f(\phi)) = \Sigma_f(\phi_{\mathrm{Un}_K})$ and that $\mathrm{Un}_{K'} \circ \Sigma_f = \Sigma_f \circ \mathrm{Un}_K$ also holds on morphisms.

Item 7: Interaction With Dependent Products

Indeed, we have

$$\begin{aligned}
 \text{Un}_{K'}(\Pi_f(X)) &\stackrel{\text{def}}{=} \prod_{x \in K'} \Pi_f(X)_x \\
 &\cong \prod_{x \in K'} \prod_{y \in f^{-1}(x)} X_y \\
 &\cong \left\{ (x, h) \in \prod_{x \in K'} \text{Sets}(f^{-1}(x), \phi_{\text{Un}_K}^{-1}(f^{-1}(x))) \mid \phi \circ h = \text{id}_{f^{-1}(x)} \right\} \\
 &\stackrel{\text{def}}{=} \Pi_f \left(\prod_{y \in K} X_y \right) \\
 &\stackrel{\text{def}}{=} \Pi_f(\text{Un}_K(X))
 \end{aligned}$$

for each $X \in \text{Obj}(\text{ISets}(K))$, where we have used **Item 2** of **Proposition 2.4.3** for the first bijection. Similarly, it can be shown that we also have $\text{Un}_{K'}(\Pi_f(\phi)) = \Pi_f(\phi_{\text{Un}_K})$ and that $\text{Un}_{K'} \circ \Pi_f = \Pi_f \circ \text{Un}_K$ also holds on morphisms. ▢

3.3 The Un/Straightening Equivalence

THEOREM 3.3.1 ► UN/STRAIGHTENING FOR INDEXED AND FIBRED SETS

We have an isomorphism of categories

$$(\text{St}_K \dashv \text{Un}_K): \text{FibSets}(K) \begin{array}{c} \xrightarrow{\text{St}_K} \\ \perp \\ \xleftarrow{\text{Un}_K} \end{array} \text{ISets}(K).$$

PROOF 3.3.2 ► PROOF OF THEOREM 3.3.1

Omitted. ▢

Appendices

A Miscellany

A.1 Other Kinds of Un/Straightening

REMARK A.1.1 ► OTHER KINDS OF UN/STRAIGHTENING

There are also other kinds of un/straightening for sets, where **Sets** is replaced by **Rel** or **Span**:

- *Un/Straightening With **Rel**, I.* We have an isomorphism of sets

$$\text{Rel}(A, B) \cong \text{Sets}(B \times A, \{\text{true}, \text{false}\}).$$

- *Un/Straightening With **Rel**, II.* We have an equivalence of categories

$$\text{LaxFun}(K_{\text{disc}}, \mathbf{Rel}) \stackrel{\text{eq.}}{\cong} \text{Cats}_{/K_{\text{disc}}}^{\text{fth}},$$

where $\text{Cats}_{/K_{\text{disc}}}^{\text{fth}}$ is the full subcategory of $\text{Cats}_{/K_{\text{disc}}}$ spanned by the faithful functors.

- *Un/Straightening With **Span**, I.* We have an isomorphism of sets

$$\text{Span}(A, B) \cong \text{Sets}(A \times B, \mathbb{N} \cup \{\infty\}).$$

- *Un/Straightening With **Span**, II.* We have an equivalence of categories

$$\text{LaxFun}(K_{\text{disc}}, \text{Span}) \stackrel{\text{eq.}}{\cong} \text{Cats}_{/K_{\text{disc}}}.$$

B Other Chapters

Logic and Model Theory

1. Logic
2. Model Theory

Type Theory

3. Type Theory
4. Homotopy Type Theory

Set Theory

5. Sets
6. Constructions With Sets
7. Indexed and Fibred Sets
8. Relations
9. Posets

Category Theory

10. Categories
11. Constructions With Categories

- 12. Limits and Colimits
- 13. Ends and Coends
- 14. Kan Extensions
- 15. Fibred Categories
- 16. Weighted Category Theory

Categorical Hochschild Co/Homology

- 17. Abelian Categorical Hochschild Co/Homology
- 18. Categorical Hochschild Co/Homology

Monoidal Categories

- 19. Monoidal Categories
- 20. Monoidal Fibrations
- 21. Modules Over Monoidal Categories
- 22. Monoidal Limits and Colimits
- 23. Monoids in Monoidal Categories
- 24. Modules in Monoidal Categories
- 25. Skew Monoidal Categories
- 26. Promonoidal Categories
- 27. 2-Groups
- 28. Duoidal Categories
- 29. Semiring Categories

Categorical Algebra

- 30. Monads
- 31. Algebraic Theories
- 32. Coloured Operads
- 33. Enriched Coloured Operads

Enriched Category Theory

- 34. Enriched Categories
- 35. Enriched Ends and Kan Extensions
- 36. Fibred Enriched Categories
- 37. Weighted Enriched Category Theory

Internal Category Theory

- 38. Internal Categories

- 39. Internal Fibrations
- 40. Locally Internal Categories
- 41. Non-Cartesian Internal Categories
- 42. Enriched-Internal Categories

Homological Algebra

- 43. Abelian Categories
- 44. Triangulated Categories
- 45. Derived Categories

Categorical Logic

- 46. Categorical Logic
- 47. Elementary Topos Theory
- 48. Non-Cartesian Topos Theory

Sites, Sheaves, and Stacks

- 49. Sites
- 50. Modules on Sites
- 51. Topos Theory
- 52. Cohomology in a Topos
- 53. Stacks

Complements on Sheaves

- 54. Sheaves of Monoids

Bicategories

- 55. Bicategories
- 56. Biadjunctions and Pseudomonads
- 57. Bilimits and Bicolimits
- 58. Biends and Bicoends
- 59. Fibred Bicategories
- 60. Monoidal Bicategories
- 61. Pseudomonoids in Monoidal Bicategories

Higher Category Theory

- 62. Tricategories
- 63. Gray Monoids and Gray Categories
- 64. Double Categories
- 65. Formal Category Theory
- 66. Enriched Bicategories

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- 67. Elementary 2-Topos Theory
 - Simplicial Stuff**
 - 68. The Simplex Category
 - 69. Simplicial Objects
 - 70. Cosimplicial Objects
 - 71. Bisimplicial Objects
 - 72. Simplicial Homotopy Theory
 - 73. Cosimplicial Homotopy Theory
 - Cyclic Stuff**
 - 74. The Cycle Category
 - 75. Cyclic Objects
 - Cubical Stuff**
 - 76. The Cube Category
 - 77. Cubical Objects
 - 78. Cubical Homotopy Theory
 - Globular Stuff**
 - 79. The Globe Category
 - 80. Globular Objects
 - Cellular Stuff**
 - 81. The Cell Category
 - 82. Cellular Objects
 - Homotopical Algebra**
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