Constructions With Sets

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Introduction

This chapter contains some material relating to constructions with sets. Notably, it contains:

- Explicit descriptions of the major types of co/limits in Sets, including in particular pushouts and coequalisers (see Definitions 1.6.1 and 1.8.1 and Remarks 1.6.2 and 1.8.2);
- A discussion of powersets as decategorifications of categories of presheaves (Remark 3.2.2);
- A lengthy discussion of the adjoint triple

  $$f_* \dashv f^{-1} \dashv f_! \colon \mathcal{P}(A) \overset{\rightarrow}{\underset{\leftarrow}{\leftarrow}} \mathcal{P}(B)$$

  of functors (morphisms of posets) between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ induced by a map of sets $f \colon A \longrightarrow B$, along with a discussion of the properties of f_*, f^{-1}, and $f_!$.
- A lengthy discussion on pointed sets and constructions with them, including in particular a discussion of the various tensor products involving them, like smash products (Section 4.6), tensors and cotensors by sets (Section 4.7), and the "left" and "right" skew tensor products of pointed sets (Section 4.8).

Notes to Myself

1. Isbell duality for powersets, $\mathrm{Lan}_{\chi}(\chi)$

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1 Limits and Colimits of Sets

1.1 Products of Families of Sets

Let $\{A_i\}_{i\in I}$ be a family of sets.

Definition 1.1.1 ▶ The Product of a Family of Sets

The **product**[1] **of** $\{A_i\}_{i\in I}$ is the set $\prod_{i\in I} A_i$ defined by

$$\prod_{i\in I} A_i \stackrel{\text{def}}{=} \left\{ f \in \mathsf{Sets}\left(I, \bigcup_{i\in I} A_i\right) \,\middle|\, \begin{array}{l}\text{for each } i \in I, \text{ we} \\ \text{have } f(i) \in A_i\end{array} \right\}.$$

[1] *Further Terminology:* Also called the **Cartesian product of** $\{A_i\}_{i\in I}$.

1.2 Coproducts of Families of Sets

Let $\{A_i\}_{i\in I}$ be a family of sets.

Definition 1.2.1 ▶ Disjoint Unions of Families

The **disjoint union of the family** $\mathcal{F}$ is the set $\coprod_{i\in I} A_i$ defined by

$$\coprod_{i\in I} A_i \stackrel{\text{def}}{=} \bigcup_{i\in I} \left\{ (x, i) \in \left(\bigcup_{i\in I} A_i\right) \times I \,\middle|\, x \in A_i \right\}.$$

1.3 Binary Products

Let A and B be sets.

Definition 1.3.1 ▶ Products of Sets

The **product**[1] **of** A **and** B is the set $A \times B$ defined by

$$\begin{aligned} A \times B &\stackrel{\text{def}}{=} \prod_{z\in\{A,B\}} z \\ &\stackrel{\text{def}}{=} \left\{ f \in \mathsf{Sets}(\{0,1\}, A \cup B) \,\middle|\, \begin{array}{l}\text{we have } f(0) \in A \\ \text{and } f(1) \in B\end{array} \right\} \end{aligned}$$

$$\cong \{\{\{a\}, \{a, b\}\} \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid \text{we have } a \in A \text{ and } b \in B\}.$$

[1] *Further Terminology:* Also called the $(\mathbb{E}_{-1}, \mathbb{E}_{-1})$**-tensor product of** A **and** B, the **Cartesian product of** A **and** B, or the **binary Cartesian product of** A **and** B, for emphasis.

Proposition 1.3.2 ▶ Properties of Products of Sets

Let A, B, C, and X be sets.

1. *Functoriality.* The assignments $A, B, (A, B) \mapsto A \times B$ define functors

$$\begin{aligned} A \times -_2 &\colon \mathsf{Sets} \to \mathsf{Sets}, \\ -_1 \times B &\colon \mathsf{Sets} \to \mathsf{Sets}, \\ -_1 \times -_2 &\colon \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets}, \end{aligned}$$

   where $-_1 \times -_2$ is the functor where

   - *Action on Objects.* For each $(A, B) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$, we have

$$[-_1 \times -_2](A, B) \overset{\text{def}}{=} A \times B;$$

   - *Action on Morphisms.* For each $(A, B), (X, Y) \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\times_{(A,B),(X,Y)} \colon \mathsf{Sets}(A, X) \times \mathsf{Sets}(B, Y) \to \mathsf{Sets}(A \times B, X \times Y)$$

     of $\times$ at $((A, B), (X, Y))$ is defined by sending (f, g) to the function

$$f \times g \colon A \times B \to X \times Y$$

     defined by

$$[f \times g](a, b) \overset{\text{def}}{=} (f(a), g(b))$$

     for each $(a, b) \in A \times B$;

   and where $A \times -$ and $- \times B$ are the partial functors of $-_1 \times -_2$ at $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

2. *Associativity.* We have an isomorphism of sets

$$(A \times B) \times C \cong A \times (B \times C),$$

   natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Unitality.* We have isomorphisms of sets

$$
\begin{aligned}
\mathrm{pt} \times A &\cong A, \\
A \times \mathrm{pt} &\cong A,
\end{aligned}
$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Commutativity.* We have an isomorphism of sets

$$A \times B \cong B \times A,$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

5. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$
\begin{aligned}
A \times \emptyset &\cong \emptyset, \\
\emptyset \times A &\cong \emptyset,
\end{aligned}
$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

6. *Distributivity Over Unions.* We have isomorphisms of sets

$$
\begin{aligned}
A \times (B \cup C) &= (A \times B) \cup (A \times C), \\
(A \cup B) \times C &= (A \times C) \cup (B \times C),
\end{aligned}
$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

7. *Distributivity Over Intersections.* We have isomorphisms of sets

$$
\begin{aligned}
A \times (B \cap C) &= (A \times B) \cap (A \times C), \\
(A \cap B) \times C &= (A \times C) \cap (B \times C),
\end{aligned}
$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

8. *Distributivity Over Differences.* We have isomorphisms of sets

$$
\begin{aligned}
A \times (B \setminus C) &= (A \times B) \setminus (A \times C), \\
(A \setminus B) \times C &= (A \times C) \setminus (B \times C),
\end{aligned}
$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

9. *Distributivity Over Symmetric Differences.* We have isomorphisms of sets

$$A \times (B \triangle C) = (A \times B) \triangle (A \times C),$$
$$(A \triangle B) \times C = (A \times C) \triangle (B \times C),$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

10. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \times, \mathrm{pt})$ is a symmetric monoidal category.

11. *Symmetric Bimonoidality.* The quintuple $(\mathsf{Sets}, \coprod, \emptyset, \times, \mathrm{pt})$ is a symmetric bimonoidal category.

Proof 1.3.3 ▶ Proof of Proposition 1.3.2

Item 1: Functoriality

Omitted.

Item 2: Associativity

Clear.

Item 3: Unitality

Clear.

Item 4: Commutativity

Clear.

Item 5: Annihilation With the Empty Set

Clear.

Item 6: Distributivity Over Unions

Omitted.

Item 7: Distributivity Over Intersections

Omitted.

Item 8: Distributivity Over Differences

Omitted.

Item 9: Distributivity Over Symmetric Differences

Omitted.

Item 10: Symmetric Monoidality

Omitted.

Item 11: Symmetric Bimonoidality

Omitted.

1.4 Binary Coproducts

Let A and B be sets.

Definition 1.4.1 ▶ Coproducts of Sets

The **coproduct**[1] **of** A **and** B is the set $A \coprod B$ defined by

$$\begin{aligned} A \coprod B &\stackrel{\text{def}}{=} \coprod_{z \in \{A,B\}} z \\ &\stackrel{\text{def}}{=} \{(a,0) \mid a \in A\} \cup \{(b,1) \mid b \in B\}. \end{aligned}$$

[1] *Further Terminology:* Also called the **disjoint union of** A **and** B, or the **binary disjoint union of** A **and** B, for emphasis.

Proposition 1.4.2 ▶ Properties of Coproducts of Sets

Let A, B, C, and X be sets.

1. *Functoriality.* The assignment $A, B, (A,B) \mapsto A \coprod B$ defines functors

$$\begin{aligned} A \coprod -_2 &\colon \mathsf{Sets} \longrightarrow \mathsf{Sets}, \\ -_1 \coprod B &\colon \mathsf{Sets} \longrightarrow \mathsf{Sets}, \\ -_1 \coprod -_2 &\colon \mathsf{Sets} \times \mathsf{Sets} \longrightarrow \mathsf{Sets}, \end{aligned}$$

where $-_1 \coprod -_2$ is the functor where

- *Action on Objects.* For each $(A,B) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$, we have

$$[-_1 \coprod -_2](A,B) \stackrel{\text{def}}{=} A \coprod B;$$

- *Action on Morphisms.* For each $(A,B), (X,Y) \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\coprod_{(A,B),(X,Y)} \colon \mathsf{Sets}(A,X) \times \mathsf{Sets}(B,Y) \longrightarrow \mathsf{Sets}(A \coprod B, X \coprod Y)$$

of $\coprod$ at $((A,B),(X,Y))$ is defined by sending (f,g) to the function

$$f \coprod g \colon A \coprod B \to X \coprod Y$$

defined by

$$[f \coprod g](x) \overset{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in A, \\ g(x) & \text{if } x \in B \end{cases}$$

for each $x \in A \coprod B$;

and where $A \coprod -$ and $- \coprod B$ are the partial functors of $-_1 \coprod -_2$ at $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

2. *Associativity.* We have an isomorphism of sets

$$(A \coprod B) \coprod C \cong A \coprod (B \coprod C),$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Unitality.* We have isomorphisms of sets

$$A \coprod \emptyset \cong A,$$
$$\emptyset \coprod A \cong A,$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Commutativity.* We have an isomorphism of sets

$$A \coprod B \cong B \coprod A,$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

5. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \coprod, \emptyset)$ is a symmetric monoidal category.

Proof 1.4.3 ▶ Proof of Proposition 1.4.2

Item 1: Functoriality

Omitted.

Item 2: Associativity

Clear.

Item 3: Unitality

Clear.

Item 4: Commutativity

Clear.

Item 5: Symmetric Monoidality

Omitted.

1.5 Pullbacks

Let A, B, and C be sets and let $f\colon A \to C$ and $g\colon B \to C$ be functions.

Definition 1.5.1 ▶ Pullbacks of Sets

The **pullback of A and B over C along f and g**[1] is the set $A \times_C B$ defined by

$$A \times_C B \overset{\text{def}}{=} \{(a, b) \in A \times B \mid f(a) = g(b)\}.$$

[1] *Further Terminology*: Also called the **fibre product of A and B over C along f and g**.

Example 1.5.2 ▶ Examples of Pullbacks of Sets

Here are some examples of pullbacks of sets.

1. *Unions via Intersections.* Let $A, B \subset X$. We have a bijection of sets

$$A \cap B \cong A \times_{A \cup B} B.$$

Proposition 1.5.3 ▶ Properties of Pullbacks of Sets

Let A, B, C, and X be sets.

1. *Associativity.* We have an isomorphism of sets

$$(A \times_X B) \times_X C \cong A \times_X (B \times_X C),$$

natural in $A, B, C, X \in \mathrm{Obj}(\mathsf{Sets})$.

2. *Unitality.* We have isomorphisms of sets

$$X \times_X A \cong A,$$
$$A \times_X X \cong A,$$

natural in $A, X \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Commutativity.* We have an isomorphism of sets

$$A \times_X B \cong B \times_X A,$$

natural in $A, B, X \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$\begin{aligned} A \times_X \emptyset &\cong \emptyset, \\ \emptyset \times_X A &\cong \emptyset, \end{aligned}$$

natural in $A, X \in \mathrm{Obj}(\mathsf{Sets})$.

5. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \times_X, X)$ is a symmetric monoidal category.

Proof 1.5.4 ▶ Proof of Proposition 1.5.3

Item 1: Associativity

Clear.

Item 2: Unitality

Clear.

Item 3: Commutativity

Clear.

Item 4: Annihilation With the Empty Set

Clear.

Item 5: Symmetric Monoidality

Omitted. □

1.6 Pushouts

Let A, B, and C be sets and let $f : C \to A$ and $g : C \to B$ be functions.

Definition 1.6.1 ▶ Pushouts of Sets

The **pushout of** A **and** B **over** C **along** f **and** g[1] is the set $A \coprod_C B$ defined by

$$A \coprod_C B \overset{\text{def}}{=} A \coprod B/{\sim_C},$$

where $\sim_C$ is the equivalence relation on $A \coprod B$ generated by $f(c) \sim_C g(c)$.

[1] *Further Terminology:* Also called the **fibre coproduct of** A **and** B **over** C **along** f **and** g.

Remark 1.6.2 ▶ Unwinding Definition 1.6.1

In detail, the relation $\sim$ of Definition 1.6.1 is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

- We have $a, b \in A$ and $a = b$;
- We have $a, b \in B$ and $a = b$;
- There exist $x_1, \ldots, x_n \in A \coprod B$ such that $a \sim' x_1 \sim' \cdots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:
  1. There exists $c \in C$ such that $x = f(c)$ and $y = g(c)$.
  2. There exists $c \in C$ such that $x = g(c)$ and $y = f(c)$.

  That is: we require the following condition to be satisfied:

  ($\star$) There exist $x_1, \ldots, x_n \in A \coprod B$ satisfying the following conditions:
    1. There exists $c_0 \in C$ satisfying one of the following conditions:
       (a) We have $a = f(c_0)$ and $x_1 = g(c_0)$.
       (b) We have $a = g(c_0)$ and $x_1 = f(c_0)$.
    2. For each $1 \leq i \leq n-1$, there exists $c_i \in C$ satisfying one of the following conditions:
       (a) We have $x_i = f(c_i)$ and $x_{i+1} = g(c_i)$.
       (b) We have $x_i = g(c_i)$ and $x_{i+1} = f(c_i)$.
    3. There exists $c_n \in C$ satisfying one of the following conditions:
       (a) We have $x_n = f(c_n)$ and $b = g(c_n)$.
       (b) We have $x_n = g(c_n)$ and $b = f(c_n)$.

Example 1.6.3 ▶ Examples of Pushouts of Sets

Here are some examples of pushouts of sets.

1. *Wedge Sums of Pointed Sets.* The wedge sum of two pointed sets of Definition 4.4.1 is an example of a pushout of sets.

2. *Intersections via Unions.* Let $A, B \subset X$. We have a bijection of sets
$$A \cup B \cong A \coprod_{A\cap B} B.$$

Proposition 1.6.4 ▶ Properties of Pushouts of Sets

Let A, B, C, and X be sets.

1. *Associativity.* We have an isomorphism of sets
$$(A \coprod_X B) \coprod_X C \cong A \coprod_X (B \coprod_X C),$$
natural in $A, B, C, X \in \mathrm{Obj}(\mathsf{Sets})$.

2. *Unitality.* We have isomorphisms of sets
$$\begin{aligned} \emptyset \coprod_X A &\cong A, \\ A \coprod_X \emptyset &\cong A, \end{aligned}$$
natural in $A, X \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Commutativity.* We have an isomorphism of sets
$$A \coprod_X B \cong B \coprod_X A,$$
natural in $A, B, X \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Annihilation With the Empty Set.* We have isomorphisms of sets
$$\begin{aligned} A \coprod_X \emptyset &\cong \emptyset, \\ \emptyset \coprod_X A &\cong \emptyset, \end{aligned}$$
natural in $A, X \in \mathrm{Obj}(\mathsf{Sets})$.

5. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \coprod_X, \emptyset)$ is a symmetric monoidal category.

Proof 1.6.5 ▶ Proof of Proposition 1.6.4

Item 1: Associativity

Clear.

Item 2: Unitality

Clear.

Item 3: Commutativity

Clear.

Item 4: Annihilation With the Empty Set

Clear.

Item 5: Symmetric Monoidality

Omitted.

1.7 Equalisers

Let A and B be sets and let $f, g\colon A \rightrightarrows B$ be functions.

Definition 1.7.1 ▶ Equalisers of Sets

The **equaliser of** f **and** g is the set $\mathrm{Eq}(f, g)$ defined by

$$\mathrm{Eq}(f, g) \overset{\text{def}}{=} \{a \in A \mid f(a) = g(a)\}.$$

Proposition 1.7.2 ▶ Properties of Equalisers of Sets

Let A, B, and C be sets.

1. *Associativity.* We have an isomorphism of sets[1]

$$\underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(g, h), g \circ \mathrm{eq}(g, h))}_{=\mathrm{Eq}(f \circ \mathrm{eq}(g,h), h \circ \mathrm{eq}(g,h))} \cong \mathrm{Eq}(f, g, h) \cong \underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(f, g), h \circ \mathrm{eq}(f, g))}_{=\mathrm{Eq}(g \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))},$$

   where $\mathrm{Eq}(f, g, h)$ is the limit of the diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \end{tikzcd}$$

   in Sets.

2. *Unitality.* We have an isomorphism of sets

$$\mathrm{Eq}(f, f) \cong A.$$

3. *Commutativity.* We have an isomorphism of sets
$$\mathrm{Eq}(f,g) \cong \mathrm{Eq}(g,f).$$
4. *Interaction With Composition.* Let
$$A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C$$
be functions. We have an inclusion of sets
$$\mathrm{Eq}(h \circ f \circ \mathrm{eq}(f,g), k \circ g \circ \mathrm{eq}(f,g)) \subset \mathrm{Eq}(h \circ f, k \circ g),$$
where $\mathrm{Eq}(h \circ f \circ \mathrm{eq}(f,g), k \circ g \circ \mathrm{eq}(f,g))$ is the equaliser of the composition
$$\mathrm{Eq}(f,g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C.$$

---

[1] That is: the following constructions give the same result:

(a) Take the equaliser of (f, g, h), i.e. the limit of the diagram
$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \end{tikzcd}$$
in Sets.

(b) First take the equaliser of f and g, forming a diagram
$$\mathrm{Eq}(f,g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B$$
and then take the equaliser of the composition
$$\mathrm{Eq}(f,g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \underset{h}{\overset{f}{\rightrightarrows}} B,$$
obtaining a subset
$$\mathrm{Eq}(f \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g)) = \mathrm{Eq}(g \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))$$
of $\mathrm{Eq}(f,g)$.

(c) First take the equaliser of g and h, forming a diagram
$$\mathrm{Eq}(g,h) \overset{\mathrm{eq}(g,h)}{\hookrightarrow} A \underset{h}{\overset{g}{\rightrightarrows}} B$$
and then take the equaliser of the composition
$$\mathrm{Eq}(g,h) \overset{\mathrm{eq}(g,h)}{\hookrightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B,$$
obtaining a subset
$$\mathrm{Eq}(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h)) = \mathrm{Eq}(f \circ \mathrm{eq}(g,h), h \circ \mathrm{eq}(g,h))$$
of $\mathrm{Eq}(g,h)$.

Proof 1.7.3 ▶ Proof of Proposition 1.7.2

Item 1: Associativity

Clear.

Item 2: Unitality

Clear.

Item 3: Commutativity

Clear.

Item 4: Interaction With Composition

Omitted. ∎

1.8 Coequalisers

Let A and B be sets and let $f, g \colon A \rightrightarrows B$ be functions.

Definition 1.8.1 ▶ Coequalisers of Sets

The **coequaliser of f and g** is the set $\mathrm{CoEq}(f, g)$ defined by

$$\mathrm{CoEq}(f, g) \overset{\text{def}}{=} B/{\sim},$$

where $\sim$ is the equivalence relation on B generated by $f(a) \sim g(a)$.

Remark 1.8.2 ▶ Unwinding Definition 1.8.1

In detail, the relation $\sim$ of Definition 1.8.1 is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

- We have $a = b$;
- There exist $x_1, \ldots, x_n \in B$ such that $a \sim' x_1 \sim' \cdots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:
  1. There exists $z \in A$ such that $x = f(z)$ and $y = g(z)$.
  2. There exists $z \in A$ such that $x = g(z)$ and $y = f(z)$.

  That is: we require the following condition to be satisfied:

  ($\star$) There exist $x_1, \ldots, x_n \in B$ satisfying the following conditions:

1. There exists $z_0 \in A$ satisfying one of the following conditions:
   (a) We have $a = f(z_0)$ and $x_1 = g(z_0)$.
   (b) We have $a = g(z_0)$ and $x_1 = f(z_0)$.
2. For each $1 \leq i \leq n-1$, there exists $z_i \in A$ satisfying one of the following conditions:
   (a) We have $x_i = f(z_i)$ and $x_{i+1} = g(z_i)$.
   (b) We have $x_i = g(z_i)$ and $x_{i+1} = f(z_i)$.
3. There exists $z_n \in A$ satisfying one of the following conditions:
   (a) We have $x_n = f(z_n)$ and $b = g(z_n)$.
   (b) We have $x_n = g(z_n)$ and $b = f(z_n)$.

Example 1.8.3 ▶ Examples of Coequalisers of Sets

Here are some examples of coequalisers of sets.

1. *Quotients by Equivalence Relations.* Let R be an equivalence relation on a set X. We have a bijection of sets

$$X/\sim_R \cong \mathrm{CoEq}\left(R \hookrightarrow X \times X \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} X\right).$$

Proposition 1.8.4 ▶ Properties of Coequalisers of Sets

Let A, B, and C be sets.

1. *Associativity.* We have an isomorphism of sets[1]

$$\underbrace{\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ f, \mathrm{coeq}(f,g)\circ h)}_{=\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ g, \mathrm{coeq}(f,g)\circ h)} \cong \mathrm{CoEq}(f,g,h) \cong \underbrace{\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f, \mathrm{coeq}(g,h)\circ g)}_{=\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f, \mathrm{coeq}(g,h)\circ h)},$$

where $\mathrm{CoEq}(f,g,h)$ is the colimit of the diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \end{tikzcd}$$

in Sets.

2. *Unitality.* We have an isomorphism of sets

$$\mathrm{CoEq}(f,f) \cong B.$$

3. *Commutativity.* We have an isomorphism of sets

$$\mathrm{CoEq}(f,g) \cong \mathrm{CoEq}(g,f).$$

4. *Interaction With Composition.* Let

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C$$

be functions. We have a surjection

$$\mathrm{CoEq}(h \circ f, k \circ g) \twoheadrightarrow \mathrm{CoEq}(\mathrm{coeq}(h,k) \circ h \circ f, \mathrm{coeq}(h,k) \circ k \circ g)$$

exhibiting $\mathrm{CoEq}(\mathrm{coeq}(h,k) \circ h \circ f, \mathrm{coeq}(h,k) \circ k \circ g)$ as a quotient of $\mathrm{CoEq}(h \circ f, k \circ g)$ by the relation generated by declaring $h(y) \sim k(y)$ for each $y \in B$.

---

[1]That is: the following constructions give the same result:

(a) Take the coequaliser of (f,g,h), i.e. the colimit of the diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \end{tikzcd}$$

in Sets.

(b) First take the coequaliser of f and g, forming a diagram

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(f,g)}{\twoheadrightarrow} \mathrm{CoEq}(f,g)$$

and then take the coequaliser of the composition

$$A \underset{h}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(f,g)}{\twoheadrightarrow} \mathrm{CoEq}(f,g),$$

obtaining a quotient

$$\mathrm{CoEq}(\mathrm{coeq}(f,g) \circ f, \mathrm{coeq}(f,g) \circ h) = \mathrm{CoEq}(\mathrm{coeq}(f,g) \circ g, \mathrm{coeq}(f,g) \circ h)$$

of $\mathrm{CoEq}(f,g)$

(c) First take the coequaliser of g and h, forming a diagram

$$A \underset{h}{\overset{g}{\rightrightarrows}} B \overset{\mathrm{coeq}(g,h)}{\twoheadrightarrow} \mathrm{CoEq}(g,h)$$

and then take the coequaliser of the composition

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(g,h)}{\twoheadrightarrow} \mathrm{CoEq}(g,h),$$

obtaining a quotient

$$\mathrm{CoEq}(\mathrm{coeq}(g,h) \circ f, \mathrm{coeq}(g,h) \circ g) = \mathrm{CoEq}(\mathrm{coeq}(g,h) \circ f, \mathrm{coeq}(g,h) \circ h)$$

of $\mathrm{CoEq}(g,h)$.

Proof 1.8.5 ▶ Proof of Proposition 1.8.4

Item 1: Associativity

Omitted.

Item 2: Unitality

Clear.

Item 3: Commutativity

Clear.

Item 4: Interaction With Composition

Omitted. □

2 Operations With Sets

2.1 The Empty Set

Definition 2.1.1 ▶ The Empty Set

The **empty set** is the set $\emptyset$ defined by

$$\emptyset \overset{\text{def}}{=} \{x \in X \mid x \neq x\},$$

where A is the set in the set existence axiom, **??** of **??**.

2.2 Singleton Sets

Let X be a set.

Definition 2.2.1 ▶ Singleton Sets

The **singleton set containing** X is the set $\{X\}$ defined by

$$\{X\} \overset{\text{def}}{=} \{X, X\},$$

where $\{X, X\}$ is the pairing of X with itself (Definition 2.3.1).

2.3 Pairings of Sets

Let X and Y be sets.

Definition 2.3.1 ▶ Pairings of Sets

The **pairing of** X **and** Y is the set $\{X, Y\}$ defined by

$$\{X, Y\} \stackrel{\text{def}}{=} \{x \in A \mid x = X \text{ or } x = Y\},$$

where A is the set in the axiom of pairing, **??** of **??**.

2.4 Unions of Families

Let $\{A_i\}_{i \in I}$ be a family of sets.

Definition 2.4.1 ▶ Unions of Families

The **union of the family** $\{A_i\}_{i \in I}$ is the set $\bigcup_{i \in I} A_i$ defined by

$$\bigcup_{i \in I} A_i \stackrel{\text{def}}{=} \{x \in F \mid \text{there exists some } i \in I \text{ such that } x \in A_i\},$$

where F is the set in the axiom of union, **??** of **??**.

2.5 Binary Unions

Let A and B be sets.

Definition 2.5.1 ▶ Binary Unions

The **union**[1] **of** A **and** B is the set $A \cup B$ defined by

$$A \cup B \stackrel{\text{def}}{=} \bigcup_{z \in \{A, B\}} z.$$

[1] *Further Terminology*: Also called the **binary union of** A **and** B, for emphasis.

Proposition 2.5.2 ▶ Properties of Binary Unions

Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cup V$ define functors

$$\begin{aligned} U \cup -\colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ - \cup V\colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \end{aligned}$$

$$-_1 \cup -_2 \colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) \to (\mathcal{P}(X), \subset),$$

where $-_1 \cup -_2$ is the functor where

- *Action on Objects.* For each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

$$[-_1 \cup -_2](U, V) \stackrel{\text{def}}{=} U \cup V;$$

- *Action on Morphisms.* For each pair of morphisms

$$\iota_U \colon U \hookrightarrow U',$$
$$\iota_V \colon V \hookrightarrow V'$$

of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

$$\iota_U \cup \iota_V \colon U \cup V \hookrightarrow U' \cup V'$$

of (ι_U, ι_V) by $\cup$ is the inclusion

$$U \cup V \subset U' \cup V'$$

i.e. where we have

  - If $U \subset U'$ and $V \subset V'$, then $U \cup V \subset U' \cup V'$;

and where $U \cup -$ and $- \cup V$ are the partial functors of $-_1 \cup -_2$ at $U, V \in \mathcal{P}(X)$.

2. *Via Intersections and Symmetric Differences.* We have an equality of sets

$$U \cup V = (U \mathbin{\triangle} V) \mathbin{\triangle} (U \cap V)$$

for each $X \in \text{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Associativity.* We have an equality of sets

$$(U \cup V) \cup W = U \cup (V \cup W)$$

for each $X \in \text{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

4. *Unitality.* We have equalities of sets

$$U \cup \emptyset = U,$$
$$\emptyset \cup U = U$$

for each $X \in \text{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

5. *Commutativity.* We have an equality of sets

$$U \cup V = V \cup U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

6. *Idempotency.* We have an equality of sets

$$U \cup U = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

7. *Distributivity Over Intersections.* We have equalities of sets

$$\begin{aligned} U \cup (V \cap W) &= (U \cup V) \cap (U \cup W), \\ (U \cap V) \cup W &= (U \cup W) \cap (V \cup W) \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

8. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

Proof 2.5.3 ▶ Proof of Proposition 2.5.2

Item 1: Functoriality

Omitted.

Item 2: Via Intersections and Symmetric Differences

Omitted.

Item 3: Associativity

Clear.

Item 4: Unitality

Clear.

Item 5: Commutativity

Clear.

Item 6: Idempotency

Clear.

Item 7: Distributivity Over Intersections

Omitted.

Item 8: Interaction With Powersets and Semirings

This follows from Items 3 to 6 and Items 3 to 5, 7 and 8 of Proposition 2.7.2.

2.6 Intersections of Families

Let $\mathcal{F}$ be a family of sets.

Definition 2.6.1 ▶ Intersections of Families

The **intersection of a family $\mathcal{F}$ of sets** is the set $\bigcap_{X\in\mathcal{F}} X$ defined by

$$\bigcap_{X\in\mathcal{F}} X \overset{\text{def}}{=} \left\{ z \in \bigcup_{X\in\mathcal{F}} X \;\middle|\; \text{for each } X \in \mathcal{F}, \text{ we have } z \in X \right\}.$$

2.7 Binary Intersections

Let X and Y be sets.

Definition 2.7.1 ▶ Binary-Intersections

The **intersection**[1] **of X and Y** is the set $X \cap Y$ defined by

$$X \cap Y \overset{\text{def}}{=} \bigcap_{z\in\{X,Y\}} z.$$

[1] *Further Terminology:* Also called the **binary intersection of X and Y**, for emphasis.

Proposition 2.7.2 ▶ Properties of Binary Intersections

Let X be a set.

1. *Functoriality.* The assignments $U, V, (U,V) \mapsto U \cap V$ define functors

$$\begin{aligned} U \cap -\colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ - \cap V\colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ -_1 \cap -_2\colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) &\to (\mathcal{P}(X), \subset), \end{aligned}$$

where $-_1 \cap -_2$ is the functor where

- *Action on Objects.* For each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

$$[-_1 \cap -_2](U, V) \overset{\text{def}}{=} U \cap V;$$

- *Action on Morphisms.* For each pair of morphisms

$$\iota_U : U \hookrightarrow U',$$
$$\iota_V : V \hookrightarrow V'$$

of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

$$\iota_U \cap \iota_V : U \cap V \hookrightarrow U' \cap V'$$

of (ι_U, ι_V) by $\cap$ is the inclusion

$$U \cap V \subset U' \cap V'$$

i.e. where we have

  - If $U \subset U'$ and $V \subset V'$, then $U \cap V \subset U' \cap V'$;

and where $U \cap -$ and $- \cap V$ are the partial functors of $-_1 \cap -_2$ at $U, V \in \mathcal{P}(X)$.

2. *Adjointness.* We have adjunctions

$$\left(U \cap - \dashv \mathbf{Hom}_{\mathcal{P}(X)}(U, -)\right)\colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "U\cap-"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X), \arrow[l, bend left, "\mathbf{Hom}_{\mathcal{P}(X)}(U,-)"] \end{tikzcd}$$

$$\left(- \cap V \dashv \mathbf{Hom}_{\mathcal{P}(X)}(V, -)\right)\colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "-\cap V"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X), \arrow[l, bend left, "\mathbf{Hom}_{\mathcal{P}(X)}(V,-)"] \end{tikzcd}$$

where

$$\mathbf{Hom}_{\mathcal{P}(X)}(-_1, -_2) : \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \to \mathcal{P}(X)$$

is the bifunctor defined by[1]

$$\mathbf{Hom}_{\mathcal{P}(X)}(U, V) \overset{\text{def}}{=} (X \setminus U) \cup V$$

witnessed by bijections

$$\mathrm{Hom}_{\mathcal{P}(X)}(U \cap V, W) \cong \mathrm{Hom}_{\mathcal{P}(X)}\big(U, \mathbf{Hom}_{\mathcal{P}(X)}(V, W)\big),$$
$$\mathrm{Hom}_{\mathcal{P}(X)}(U \cap V, W) \cong \mathrm{Hom}_{\mathcal{P}(X)}\big(V, \mathbf{Hom}_{\mathcal{P}(X)}(U, W)\big),$$

natural in $U, V, W \in \mathcal{P}(X)$, i.e. where:

(a) The following conditions are equivalent:
   (i) We have $U \cap V \subset W$.
   (ii) We have $U \subset \mathbf{Hom}_{\mathcal{P}(X)}(V, W)$.
   (iii) We have $U \subset (X \setminus V) \cup W$.

(b) The following conditions are equivalent:
   (i) We have $V \cap U \subset W$.
   (ii) We have $V \subset \mathbf{Hom}_{\mathcal{P}(X)}(U, W)$.
   (iii) We have $V \subset (X \setminus U) \cup W$.

3. *Associativity.* We have an equality of sets

$$(U \cap V) \cap W = U \cap (V \cap W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

4. *Unitality.* Let X be a set and let $U \in \mathcal{P}(X)$. We have equalities of sets

$$\begin{aligned} X \cap U &= U, \\ U \cap X &= U \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

5. *Commutativity.* We have an equality of sets

$$U \cap V = V \cap U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

6. *Idempotency.* We have an equality of sets

$$U \cap U = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

7. *Distributivity Over Unions.* We have equalities of sets

$$\begin{aligned} U \cap (V \cup W) &= (U \cap V) \cup (U \cap W), \\ (U \cup V) \cap W &= (U \cap W) \cup (V \cap W) \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

8. *Annihilation With the Empty Set.* We have an equality of sets

$$\emptyset \cap X = \emptyset,$$
$$X \cap \emptyset = \emptyset$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

9. *Interaction With Powersets and Monoids With Zero.* The quadruple $((\mathcal{P}(X), \emptyset), \cap, X)$ is a commutative monoid with zero.

10. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

[1] *Intuition:* Since intersections are the products in $\mathcal{P}(X)$, the left adjoint $\mathbf{Hom}_{\mathcal{P}(X)}(U, V)$ works as a function type $U \to V$.

Now, under the Curry–Howard correspondence, the function type $U \to V$ corresponds to implication $U \implies V$, which is logically equivalent to the statement $\neg U \vee V$, which in turn corresponds to the set $U^{\mathsf{c}} \vee V \overset{\text{def}}{=} (X \setminus U) \cup V$.

Proof 2.7.3 ▶ Proof of Proposition 2.7.2

Item 1: Functoriality

Omitted.

Item 2: Adjointness

See [MSE 267469].

Item 3: Associativity

Clear.

Item 4: Unitality

Clear.

Item 5: Commutativity

Clear.

Item 6: Idempotency

Clear.

Item 7: Distributivity Over Unions

Omitted.

Item 8: Annihilation With the Empty Set

Clear.

Item 9: Interaction With Powersets and Monoids With Zero

This follows from Items 3 to 5 and 8.

Item 10: Interaction With Powersets and Semirings

This follows from Items 3 to 6 and Items 3 to 5, 7 and 8 of Proposition 2.7.2.

2.8 Differences

Let X and Y be sets.

Definition 2.8.1 ▶ Differences

The **difference of X and Y** is the set $X \setminus Y$ defined by

$$X \setminus Y \overset{\text{def}}{=} \{a \in X \mid a \notin Y\}.$$

Proposition 2.8.2 ▶ Properties of Differences

Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cap V$ define functors

$$\begin{aligned} U \setminus -\colon (\mathcal{P}(X), \supset) &\to (\mathcal{P}(X), \subset),\\ - \setminus V\colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset),\\ -_1 \setminus -_2\colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \supset) &\to (\mathcal{P}(X), \subset), \end{aligned}$$

where $-_1 \setminus -_2$ is the functor where

- *Action on Objects.* For each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

$$[-_1 \setminus -_2](U, V) \overset{\text{def}}{=} U \setminus V;$$

- *Action on Morphisms.* For each pair of morphisms

$$\begin{aligned} \iota_A &\colon A \hookrightarrow B,\\ \iota_U &\colon U \hookrightarrow V \end{aligned}$$

of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

$$\iota_U \setminus \iota_V \colon A \setminus V \hookrightarrow B \setminus U$$

of (ι_U, ι_V) by $\setminus$ is the inclusion

$$A \setminus V \subset B \setminus U$$

i.e. where we have

- If $A \subset B$ and $U \subset V$, then $A \setminus V \subset B \setminus U$;

and where $U \setminus -$ and $- \setminus V$ are the partial functors of $-_1 \setminus -_2$ at $U, V \in \mathcal{P}(X)$.

2. *De Morgan's Laws*. We have equalities of sets

$$\begin{aligned} X \setminus (U \cup V) &= (X \setminus U) \cap (X \setminus V), \\ X \setminus (U \cap V) &= (X \setminus U) \cup (X \setminus V) \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Interaction With Unions*. We have equalities of sets

$$(U \setminus V) \cup W = (U \cup W) \setminus (V \setminus W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

4. *Interaction With Intersections*. We have equalities of sets

$$\begin{aligned} (U \setminus V) \cap W &= (U \cap W) \setminus V \\ &= U \cap (W \setminus V) \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

5. *Triple Differences*. We have

$$U \setminus (V \setminus W) = (U \cap W) \cup (U \setminus V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

6. *Left Annihilation*. We have

$$\emptyset \setminus U = \emptyset$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

7. *Right Unitality.* We have

$$U \setminus \emptyset = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

8. *Invertibility.* We have

$$U \setminus U = \emptyset$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

9. *Interaction With Containment.* The following conditions are equivalent:

   (a) We have $V \setminus U \subset W$.

   (b) We have $V \setminus W \subset U$.

Proof 2.8.3 ▶ Proof of Proposition 2.8.2

Item 1: Functoriality

Omitted.

Item 2: De Morgan's Laws

Omitted.

Item 3: Interaction With Unions

Omitted.

Item 4: Interaction With Intersections

Omitted.

Item 5: Triple Differences

Omitted.

Item 6: Left Annihilation

Clear.

Item 7: Right Unitality

Clear.

Item 8: Invertibility

Clear.

Item 9: Interaction With Containment

Omitted. □

2.9 Complements

Let X be a set and let $U \in \mathcal{P}(X)$.

Definition 2.9.1 ▶ Complements

The **complement of** U is the set U^{c} defined by

$$\begin{aligned} U^{\mathsf{c}} &\overset{\mathrm{def}}{=} X \setminus U \\ &\overset{\mathrm{def}}{=} \{a \in X \mid a \notin U\}. \end{aligned}$$

Proposition 2.9.2 ▶ Properties of Complements

Let X be a set.

1. *Functoriality.* The assignment $U \mapsto U^{\mathsf{c}}$ defines a functor

   $$(-)^{\mathsf{c}} \colon \mathcal{P}(X)^{\mathsf{op}} \to \mathcal{P}(X),$$

   where

   - *Action on Objects.* For each $U \in \mathcal{P}(X)$, we have

     $$\left[(-)^{\mathsf{c}}\right](U) \overset{\mathrm{def}}{=} U^{\mathsf{c}};$$

   - *Action on Morphisms.* For each morphism $\iota_U \colon U \hookrightarrow V$ of $\mathcal{P}(X)$, the image

     $$\iota_U^{\mathsf{c}} \colon V^{\mathsf{c}} \hookrightarrow U^{\mathsf{c}}$$

     of ι_U by $(-)^{\mathsf{c}}$ is the inclusion

     $$V^{\mathsf{c}} \subset U^{\mathsf{c}}$$

     i.e. where we have
     - If $U \subset V$, then $V^{\mathsf{c}} \subset U^{\mathsf{c}}$.

2. *De Morgan's Laws.* We have equalities of sets

   $$\begin{aligned} (U \cup V)^{\mathsf{c}} &= U^{\mathsf{c}} \cap V^{\mathsf{c}}, \\ (U \cap V)^{\mathsf{c}} &= U^{\mathsf{c}} \cup V^{\mathsf{c}} \end{aligned}$$

   for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Involutority.* We have
$$(U^{\mathsf{c}})^{\mathsf{c}} = U$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

Proof 2.9.3 ▶ Proof of Proposition 2.9.2

Item 1: Functoriality

Clear.

Item 2: De Morgan's Laws

Omitted.

Item 3: Involutority

Clear.

2.10 Symmetric Differences

Let A and B be sets.

Definition 2.10.1 ▶ Symmetric-Differences

The **symmetric difference of** A **and** B is the set $A \bigtriangleup B$ defined by
$$A \bigtriangleup B \overset{\text{def}}{=} (A \setminus B) \cup (B \setminus A).$$

Proposition 2.10.2 ▶ Properties of Symmetric Differences

Let X be a set.

1. *Lack of Functoriality.* The assignment $(U, V) \mapsto U \bigtriangleup V$ ***does not*** define a functor
$$-_1 \bigtriangleup -_2 \colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) \to (\mathcal{P}(X), \subset).$$

2. *Via Unions and Intersections.* We have[1]
$$U \bigtriangleup V = (U \cup V) \setminus (U \cap V)$$
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

3. *Associativity*. We have[2]

$$(U \vartriangle V) \vartriangle W = U \vartriangle (V \vartriangle W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

4. *Unitality*. We have

$$\begin{aligned} U \vartriangle \emptyset &= U, \\ \emptyset \vartriangle U &= U \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

5. *Invertibility*. We have

$$U \vartriangle U = \emptyset$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

6. *Commutativity*. We have

$$U \vartriangle V = V \vartriangle U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

7. "*Transitivity*". We have

$$(U \vartriangle V) \vartriangle (V \vartriangle W) = U \vartriangle W$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

8. *The Triangle Inequality for Symmetric Differences*. We have

$$U \vartriangle W \subset U \vartriangle V \cup V \vartriangle W$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

9. *Distributivity Over Intersections*. We have

$$\begin{aligned} U \cap (V \vartriangle W) &= (U \cap V) \vartriangle (U \cap W), \\ (U \vartriangle V) \cap W &= (U \cap W) \vartriangle (V \cap W) \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

10. *Interaction With Indicator Functions.* We have

$$\chi_{U \triangle V} \equiv \chi_U + \chi_V \pmod 2$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

11. *Interaction With Powersets and Groups I.* The quadruple $\left(\mathcal{P}(X), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(X)}\right)$ is an abelian group.[3,4,5]

12. *Interaction With Powersets and Groups II.* Every element of $\mathcal{P}(X)$ has order 2 with respect to $\triangle$, and thus $\mathcal{P}(X)$ is a *Boolean group* (i.e. an abelian 2-group).

13. *Interaction With Powersets and Vector Spaces I.* The pair $\left(\mathcal{P}(X), \alpha_{\mathcal{P}(X)}\right)$ consisting of

    - The group $\mathcal{P}(X)$ of Item 11;
    - The map $\alpha_{\mathcal{P}(X)} \colon \mathbb{F}_2 \times \mathcal{P}(X) \to \mathcal{P}(X)$ defined by

    $$\begin{aligned} 0 \cdot U &\overset{\text{def}}{=} \emptyset, \\ 1 \cdot U &\overset{\text{def}}{=} U; \end{aligned}$$

    is an $\mathbb{F}_2$-vector space.

14. *Interaction With Powersets and Vector Spaces II.* If X is finite, then:

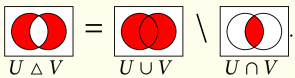
    (a) The set of singletons sets on the elements of X forms a basis for the $\mathbb{F}_2$-vector space $\left(\mathcal{P}(X), \alpha_{\mathcal{P}(X)}\right)$ of Item 13.

    (b) We have

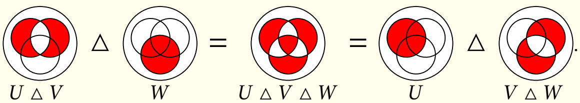
    $$\dim(\mathcal{P}(X)) = \#\mathcal{P}(X).$$

15. *Interaction With Powersets and Rings.* The quintuple $(\mathcal{P}(X), \triangle, \cap, \emptyset, X)$ is a commutative ring.[6]

[1] *Illustration* (from [Wik22]):



[2] *Illustration* (from [Wik22]):



[3] *Example:* When $X = \emptyset$, we have an isomorphism of groups between $\mathcal{P}(\emptyset)$ and the trivial group:

$$\left(\mathcal{P}(\emptyset), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\emptyset)}\right) \cong \mathrm{pt}.$$

[4] *Example:* When $X = \mathrm{pt}$, we have an isomorphism of groups between $\mathcal{P}(\mathrm{pt})$ and $\mathbb{Z}_{/2}$:

$$\left(\mathcal{P}(\mathrm{pt}), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\mathrm{pt})}\right) \cong \mathbb{Z}_{/2}.$$

[5] *Example:* When $X = \{0, 1\}$, we have an isomorphism of groups between $\mathcal{P}(\{0, 1\})$ and $\mathbb{Z}_{/2} \times \mathbb{Z}_{/2}$:

$$\left(\mathcal{P}(\{0, 1\}), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\{0,1\})}\right) \cong \mathbb{Z}_{/2} \times \mathbb{Z}_{/2}.$$

[6] *Warning:* The analogous statement replacing intersections by unions (i.e. that the quintuple $(\mathcal{P}(X), \triangle, \cup, \emptyset, X)$ is a ring) is false, however; see [Pro23b].

Proof 2.10.3 ▶ Proof of Proposition 2.10.2

Item 1: Lack of Functoriality

Omitted.

Item 2: Via Unions and Intersections

Omitted.

Item 3: Associativity

Omitted.

Item 4: Unitality

Clear.

Item 5: Invertibility

Clear.

Item 6: Commutativity

Clear.

Item 7: “Transitivity”

We have

$$\begin{aligned}
(U \triangle V) \triangle (V \triangle W) &= U \triangle (V \triangle (V \triangle W)), && \text{(Item 3)}\\
&= U \triangle ((V \triangle V) \triangle W), && \text{(Item 3)}\\
&= U \triangle (\emptyset \triangle W), && \text{(Item 5)}\\
&= U \triangle W. && \text{(Item 4)}
\end{aligned}$$

Item 8: The Triangle Inequality for Symmetric Differences

This follows from Items 2 and 7.

Item 9: Distributivity Over Intersections

Omitted.

Item 10: Interaction With Indicator Functions

Clear.

Item 11: Interaction With Powersets and Groups I

This follows from Items 3 to 6.

Item 12: Interaction With Powersets and Groups II

This follows from Item 5.

Item 13: Interaction With Powersets and Vector Spaces I

Clear.

Item 14: Interaction With Powersets and Vector Spaces II

Omitted.

Item 15: Interaction With Powersets and Rings

This follows from Items 9 and 11 and Items 8 and 9 of Proposition 2.7.2.[1]

[1] *Reference*: [Pro23a].

2.11 Ordered Pairs

Let A and B be sets.

Definition 2.11.1 ▶ Ordered-Pairs

The **ordered pair associated to A and B** is the set (A, B) defined by

$$(A, B) \overset{\text{def}}{=} \{\{A\}, \{A, B\}\}.$$

Proposition 2.11.2 ▶ Properties of Ordered Pairs

Let A and B be sets.

1. *Uniqueness*. Let A, B, C, and D be sets. The following conditions are equivalent:

   (a) We have $(A, B) = (C, D)$.

   (b) We have $A = C$ and $B = D$.

PROOF 2.11.3 ▶ PROOF OF PROPOSITION 2.11.2

Item 1: Uniqueness

See [Cie97, Theorem 1.2.3].

3 Powersets

3.1 Characteristic Functions

Let X be a set.

DEFINITION 3.1.1 ▶ CHARACTERISTIC FUNCTIONS

Let $U \subset X$ and let $x \in X$.

1. The **characteristic function of** U is the function[1]

$$\chi_U \colon X \longrightarrow \{\text{true}, \text{false}\}$$

   defined by

$$\chi_U(x) \overset{\text{def}}{=} \begin{cases} \text{true} & \text{if } x \in U, \\ \text{false} & \text{if } x \notin U \end{cases}$$

   for each $x \in X$.

2. The **characteristic function of** x is the function[2]

$$\chi_x \colon X \longrightarrow \{\text{true}, \text{false}\}$$

   defined by

$$\chi_x \overset{\text{def}}{=} \chi_{\{x\}},$$

   i.e. by

$$\chi_x(y) \overset{\text{def}}{=} \begin{cases} \text{true} & \text{if } x = y, \\ \text{false} & \text{if } x \neq y \end{cases}$$

   for each $y \in X$.

3. The **characteristic relation on** X[3] is the relation[4]

$$\chi_X(-_1, -_2) \colon X \times X \longrightarrow \{\text{true}, \text{false}\}$$

on X defined by[5]

$$\chi_X(x,y) \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } x = y, \\ \text{false} & \text{if } x \neq y \end{cases}$$

for each $x, y \in X$.

4. The **characteristic embedding**[6] **of X into $\mathcal{P}(X)$** is the function

$$\chi_{(-)} \colon X \hookrightarrow \mathcal{P}(X)$$

defined by

$$\chi_{(-)}(x) \stackrel{\text{def}}{=} \chi_x$$

for each $x \in X$.

[1] *Further Notation:* Also written $\chi_X(U,-)$ or $\chi_X(-,U)$.
[2] *Further Notation:* Also written χ_x, $\chi_X(x,-)$, or $\chi_X(-,x)$.
[3] *Further Terminology:* Also called the **identity relation on X**.
[4] *Further Notation:* Also written $\chi^{-1}_{-_2}$, or $\sim_{\text{id}}$ in the context of relations.
[5] As a set, χ_X corresponds to the diagonal $\Delta_X \subset X \times X$ of X.
[6] The name "characteristic *embedding*" comes from the fact that there is an analogue of fully faithfulness for $\chi_{(-)}$: given a set X, we have

$$\chi^{\text{Pos}}_{\mathcal{P}(X)}(\chi_x, \chi_y) = \chi_X(x,y),$$

for each $x, y \in X$, where $\chi^{\text{Pos}}_{\mathcal{P}(X)}$ is the *posetal characteristic relation of $\mathcal{P}(X)$* of **??**.

Remark 3.1.2 ▶ Characteristic-Functions-as-Decategorifications-of-Presheaves

The definitions in Definition 3.1.1 are decategorifications of co/presheaves, representable co/presheaves, Hom profunctors, and the Yoneda embedding:[1]

1. A function
$$f \colon X \longrightarrow \{\text{true}, \text{false}\}$$
is a decategorification of a presheaf
$$\mathcal{F} \colon \mathcal{C}^{\text{op}} \longrightarrow \text{Sets},$$
with the characteristic functions χ_U of the subsets of X being the primordial examples (and, in fact, all examples) of these;

2. The characteristic function
$$\chi_x \colon X \longrightarrow \{\text{true}, \text{false}\}$$

of an *element* x of X is a decategorification of the representable presheaf

$$h_X : \mathcal{C}^{\mathsf{op}} \to \mathsf{Sets}$$

of an *object* x of a category $\mathcal{C}$;

3. The characteristic relation

$$\chi_X(-_1, -_2) : X \times X \to \{\mathsf{true}, \mathsf{false}\}$$

of X is a decategorification of the Hom profunctor

$$\mathsf{Hom}_{\mathcal{C}}(-_1, -_2) : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Sets}$$

of a category $\mathcal{C}$;

4. The characteristic embedding

$$\chi_{(-)} : X \hookrightarrow \mathcal{P}(X)$$

of X into $\mathcal{P}(X)$ is a decategorification of the Yoneda embedding

$$よ : \mathcal{C}^{\mathsf{op}} \hookrightarrow \mathsf{PSh}(\mathcal{C})$$

of a category $\mathcal{C}$ into $\mathsf{PSh}(\mathcal{C})$;

5. Unions and colimits:
   - An element of $\mathcal{P}(X)$ is a union of elements of X, viewed as one-point subsets $\{x\} \in \mathcal{P}(A)$;
   - An object of $\mathsf{PSh}(\mathcal{C})$ is a colimit of objects of $\mathcal{C}$, viewed as representable presheaves $h_X \in \mathsf{Obj}(\mathsf{PSh}(\mathcal{C}))$.

[1] These statements can be made precise by using the embeddings

$$(-)_{\mathsf{disc}} : \mathsf{Sets} \hookrightarrow \mathsf{Cats},$$
$$(-)_{\mathsf{disc}} : \{\mathsf{true}, \mathsf{false}\}_{\mathsf{disc}} \hookrightarrow \mathsf{Sets}$$

of sets into categories and of classical truth values into sets.

For instance, in this approach the characteristic function

$$\chi_x : X \to \{\mathsf{true}, \mathsf{false}\}$$

of an element x of X, defined by

$$\chi_x(y) \overset{\mathrm{def}}{=} \begin{cases} \mathsf{true} & \text{if } x = y, \\ \mathsf{false} & \text{if } x \neq y \end{cases}$$

for each $y \in X$, is recovered as the representable presheaf

$$\mathsf{Hom}_{X_{\mathsf{disc}}}(-, x) : X_{\mathsf{disc}} \to \mathsf{Sets}$$

of the corresponding object x of X_{disc}, defined on objects by

$$\mathsf{Hom}_{X_{\mathsf{disc}}}(y, x) \overset{\mathrm{def}}{=} \begin{cases} \mathsf{pt} & \text{if } x = y, \\ \emptyset & \text{if } x \neq y \end{cases}$$

for each $y \in \mathsf{Obj}(X_{\mathsf{disc}})$.

Proposition 3.1.3 ▶ The Yoneda Lemma for Sets

Let X be a set and let $U \subset X$ be a subset of X. We have

$$\chi^{\mathsf{Pos}}_{\mathcal{P}(X)}(\chi_x, \chi_U) = \chi_U(x)$$

for each $x \in X$, giving an equality of functions

$$\chi^{\mathsf{Pos}}_{\mathcal{P}(X)}\left(\chi_{(-)}, \chi_U\right) = \chi_U.$$

Proof 3.1.4 ▶ Proof of Proposition 3.1.3

Clear.

3.2 Powersets

Let X be a set.

Definition 3.2.1 ▶ Powersets

The **powerset of** X is the set $\mathcal{P}(X)$ defined by

$$\mathcal{P}(X) \stackrel{\text{def}}{=} \{U \in P \mid U \subset X\},$$

where P is the set in the axiom of powerset, **??** of **??**.

Remark 3.2.2 ▶ Powersets as Decategorifications of Co/Presheaf Categories

The powerset of a set is a decategorification of the category of presheaves of a category: while

- The powerset of a set X is equivalently (Item 6 of Proposition 3.2.3) the set

$$\mathsf{Sets}(X, \{\mathsf{true}, \mathsf{false}\})$$

of functions from X to the set $\{\mathsf{true}, \mathsf{false}\}$ of classical truth values;

- The category of presheaves on a category C is the category

$$\mathsf{Fun}(C^{\mathsf{op}}, \mathsf{Sets})$$

of functors from C^{op} to the category Sets of sets.

where we notice that while

- A category is enriched over the category

$$\mathsf{Sets} \overset{\text{def}}{=} \mathsf{Cats}_0$$

  of sets, with presheaves taking values on it;

- A set is enriched over the set

$$\{\mathsf{true}, \mathsf{false}\} \overset{\text{def}}{=} \mathsf{Cats}_{-1}$$

  of classical truth values, with characteristic functions taking values on it.

Proposition 3.2.3 ▶ Properties of Powersets

Let X be a set.

1. *Functoriality.* The assignment $X \mapsto \mathcal{P}(X)$ defines functors

$$\begin{aligned} \mathcal{P}_* &\colon \mathsf{Sets} \to \mathsf{Sets}, \\ \mathcal{P}^{-1} &\colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Sets}, \\ \mathcal{P}_! &\colon \mathsf{Sets} \to \mathsf{Sets} \end{aligned}$$

   where

   - *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\begin{aligned} \mathcal{P}_*(A) &\overset{\text{def}}{=} \mathcal{P}(A), \\ \mathcal{P}^{-1}(A) &\overset{\text{def}}{=} \mathcal{P}(A), \\ \mathcal{P}_!(A) &\overset{\text{def}}{=} \mathcal{P}(A); \end{aligned}$$

   - *Action on Morphisms.* For each morphism $f\colon A \to B$ of Sets, the images

$$\begin{aligned} \mathcal{P}_*(f) &\colon \mathcal{P}(A) \to \mathcal{P}(B), \\ \mathcal{P}^{-1}(f) &\colon \mathcal{P}(B) \to \mathcal{P}(A), \\ \mathcal{P}_!(f) &\colon \mathcal{P}(A) \to \mathcal{P}(B) \end{aligned}$$

     of f by $\mathcal{P}_*$, $\mathcal{P}^{-1}$, and $\mathcal{P}_!$ are defined by

$$\mathcal{P}_*(f) \overset{\text{def}}{=} f_*,$$

$$\mathcal{P}^{-1}(f) \stackrel{\text{def}}{=} f^{-1},$$
$$\mathcal{P}_!(f) \stackrel{\text{def}}{=} f_!,$$

as in Definitions 3.3.1, 3.4.1 and 3.5.1.

2. *Adjointness I.* We have an adjunction

$$\left(\mathcal{P}^{-1} \dashv \mathcal{P}^{-1,\mathsf{op}}\right)\colon \quad \begin{tikzcd} \mathsf{Sets}^{\mathsf{op}} \arrow[r, bend left, "\mathcal{P}^{-1}"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathcal{P}^{-1,\mathsf{op}}"] \end{tikzcd}$$

witnessed by a bijection

$$\underbrace{\mathsf{Sets}^{\mathsf{op}}(\mathcal{P}(X), Y)}_{\stackrel{\text{def}}{=}\mathsf{Sets}(Y,\mathcal{P}(X))} \cong \mathsf{Sets}(X, \mathcal{P}(Y)),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $Y \in \mathrm{Obj}\left(\mathsf{Sets}^{\mathsf{op}}\right)$.

3. *Adjointness II.* We have an adjunction

$$(\Gamma \dashv \mathcal{P}_*)\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "\Gamma"] \arrow[r, phantom, "\perp"] & \mathsf{Rel}, \arrow[l, bend left, "\mathcal{P}_*"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathsf{Rel}(\Gamma(A), B) \cong \mathsf{Sets}(A, \mathcal{P}(B))$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $B \in \mathrm{Obj}(\mathsf{Rel})$.

4. *Symmetric Strong Monoidality With Respect to Coproducts.* The powerset functor $\mathcal{P}_*$ of Item 1 has a symmetric strong monoidal structure

$$\left(\mathcal{P}_*, \mathcal{P}_*^{\coprod}, \mathcal{P}_{*|\mathbb{1}}^{\coprod}\right)\colon (\mathsf{Sets}, \coprod, \emptyset) \to (\mathsf{Sets}, \times, \mathrm{pt})$$

being equipped with isomorphisms

$$\mathcal{P}_{*|X,Y}^{\coprod}\colon \mathcal{P}(X) \times \mathcal{P}(Y) \xrightarrow{\cong} \mathcal{P}(X \coprod Y),$$
$$\mathcal{P}_{*|\mathbb{1}}^{\coprod}\colon \mathrm{pt} \xrightarrow{=} \mathcal{P}(\emptyset),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

5. *Symmetric Lax Monoidality With Respect to Products.* The powerset functor $\mathcal{P}_*$ of Item 1 has a symmetric lax monoidal structure

$$\left(\mathcal{P}_*, \mathcal{P}_*^{\otimes}, \mathcal{P}_{*|\mathbb{1}}^{\otimes}\right) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \to (\mathsf{Sets}, \times, \mathrm{pt})$$

being equipped with isomorphisms

$$\mathcal{P}_{*|X,Y}^{\otimes} \colon \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(X \times Y),$$
$$\mathcal{P}_{*|\mathbb{1}}^{\otimes} \colon \mathrm{pt} \xrightarrow{=} \mathcal{P}(\emptyset),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$, where $\mathcal{P}_{*|X,Y}^{\otimes}$ is given by

$$\mathcal{P}_{*|X,Y}^{\otimes}(U, V) \stackrel{\text{def}}{=} U \times V$$

for each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(Y)$.

6. *Powersets as Sets of Functions.* The assignment $U \mapsto \chi_U$ defines a bijection[1]

$$\chi_{(-)} \colon \mathcal{P}(X) \xrightarrow{\cong} \mathsf{Sets}(X, \{\mathrm{true}, \mathrm{false}\}),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$.

7. *Powersets as Sets of Relations.* We have bijections

$$\mathcal{P}(X) \cong \mathrm{Rel}(\mathrm{pt}, X),$$
$$\mathcal{P}(X) \cong \mathrm{Rel}(X, \mathrm{pt}),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$.

8. *As a Free Cocompletion: Universal Property.* The pair $(\mathcal{P}(X), \chi_{(-)})$ consisting of

   - The powerset $\mathcal{P}(X)$ of X;
   - The characteristic embedding $\chi_{(-)} \colon X \hookrightarrow \mathcal{P}(X)$ of X into $\mathcal{P}(X)$;

   satisfies the following universal property:

   (UP) Given another pair (Y, f) consisting of

   - A cocomplete poset $(Y, \preceq)$;
   - A function $f \colon X \to Y$;

there exists a unique cocontinuous morphism of posets $(\mathcal{P}(X), \subset) \xrightarrow{\exists!} (Y, \preceq)$ making the diagram

$$\begin{tikzcd} & \mathcal{P}(X) \arrow[d, dashed, "\exists!"] \\ X \arrow[ur, "\chi_X"] \arrow[r, "f"'] & Y \end{tikzcd}$$

commute.

9. *As a Free Cocompletion: Adjointness.* We have an adjunction[2]

$$(\chi_{(-)} \dashv 忘)\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "\chi_{(-)}"] \arrow[r, phantom, "\perp"] & \mathsf{Pos}^{\mathsf{cocomp.}}, \arrow[l, bend left, "忘"] \end{tikzcd}$$

witnessed by a bijection

$$\mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq)) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, \preceq) \in \mathrm{Obj}(\mathsf{Pos})$, where

- We have a natural map

$$\chi_X^* \colon \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq)) \to \mathsf{Sets}(X, Y)$$

defined by

$$\chi_X^*(f) \stackrel{\text{def}}{=} f \circ \chi_X,$$

i.e. by sending a cocontinuous morphism of posets $f \colon \mathcal{P}(X) \to Y$ to the composition

$$X \overset{\chi_X}{\hookrightarrow} \mathcal{P}(X) \xrightarrow{f} Y;$$

- We have a natural map

$$\mathrm{Lan}_{\chi_X} \colon \mathsf{Sets}(X, Y) \to \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq))$$

computed by

$$\left[\mathrm{Lan}_{\chi_X}(f)\right](U) \cong \int^{x \in X} \chi_{\mathcal{P}(X)}(\chi_x, U) \odot f(x)$$

$$
\begin{aligned}
&\cong \int^{x \in X} \chi_U(x) \odot f(x) && \text{(Proposition 3.1.3)} \\
&\cong \bigvee_{x \in X} (\chi_U(x) \odot f(x))
\end{aligned}
$$

for each $U \in \mathcal{P}(X)$, where:

- $\bigvee$ is the join in $(Y, \preceq)$;
- We have

$$
\begin{aligned}
\text{true} \odot f(x) &\stackrel{\text{def}}{=} f(x), \\
\text{false} \odot f(x) &\stackrel{\text{def}}{=} \varnothing_Y,
\end{aligned}
$$

where $\varnothing_Y$ is the minimal element of $(Y, \preceq)$.

[1]This bijection is a decategorified form of the equivalence

$$\mathsf{DFib}(C) \stackrel{\text{eq.}}{\cong} \mathsf{PSh}(C)$$

of Fibred Categories, **??** of Proposition 9.4.1, with $\chi_{(-)}$ being a decategorified version of the category of elements construction of Fibred Categories, Definition 9.2.1.

See also **??** of **??**.

[2]In this sense, $\mathcal{P}(A)$ is the free cocompletion of A. (Note that, despite its name, however, this is not an idempotent operation, as we have $\mathcal{P}(\mathcal{P}(A)) \neq \mathcal{P}(A)$.)

Proof 3.2.4 ▶ Proof of Proposition 3.2.3

Item 1: Functoriality

This follows from Items 3 to 4.

Item 2: Adjointness I

Omitted.

Item 3: Adjointness II

Omitted.

Item 4: Symmetric Strong Monoidality With Respect to Coproducts

Omitted.

Item 5: Symmetric Lax Monoidality With Respect to Products

Omitted.

Item 6: Powersets as Sets of Functions

Omitted.

Item 7: Powersets as Sets of Relations

Omitted.

Item 8: As a Free Cocompletion: Universal Property

This is a rephrasing of **??**.

Item 9: As a Free Cocompletion: Adjointness

Omitted.

3.3 Direct Images

Let A and B be sets and let $f\colon A \longrightarrow B$ be a function.

Definition 3.3.1 ▶ Direct Images

The **direct image function associated to** f is the function[1]

$$f_*\colon \mathcal{P}(A) \longrightarrow \mathcal{P}(B)$$

defined by[2,3]

$$\begin{aligned} f_*(U) &\stackrel{\text{def}}{=} f(U) \\ &\stackrel{\text{def}}{=} \left\{ b \in B \;\middle|\; \begin{array}{l}\text{there exists some } a \in \\ U \text{ such that } b = f(a)\end{array} \right\} \\ &= \{f(a) \in B \mid a \in U\} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

[1] *Further Notation:* Also written $\exists_f\colon \mathcal{P}(A) \to \mathcal{P}(B)$. This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

- We have $b \in \exists_f(U)$.
- There exists some $a \in U$ such that $f(a) = b$.

[2] *Further Terminology:* The set $f(U)$ is called the **direct image of** U **by** f.

[3] We also have

$$f_*(U) = B \setminus f_!(A \setminus U);$$

see Item 7 of Proposition 3.3.3.

Remark 3.3.2 ▶ Unwinding Definition 3.3.1

Identifying subsets of A with functions from A to $\{\mathsf{true}, \mathsf{false}\}$ via Item 6 of Proposition 3.2.3, we see that the direct image function associated to f is equivalently the function

$$f_* \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by

$$\begin{aligned} f_*(\chi_U) &\overset{\text{def}}{=} \mathrm{Lan}_f(\chi_U) \\ &= \mathrm{colim}\Big(\Big(f \downarrow \underline{(-_1)}\Big) \overset{\mathrm{pr}}{\twoheadrightarrow} A \xrightarrow{\chi_U} \{\mathsf{true}, \mathsf{false}\}\Big) \\ &= \operatorname*{colim}_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)) \\ &= \bigvee_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)). \end{aligned}$$

So, in other words, we have

$$\begin{aligned} [f_*(\chi_U)](b) &= \bigvee_{\substack{a \in A \\ f(a) = b}} (\chi_U(a)) \\ &= \begin{cases} \mathsf{true} & \text{if there exists some } a \in A \text{ such} \\ & \text{that } f(a) = b \text{ and } a \in U, \\ \mathsf{false} & \text{otherwise} \end{cases} \\ &= \begin{cases} \mathsf{true} & \text{if there exists some } a \in U \\ & \text{such that } f(a) = b, \\ \mathsf{false} & \text{otherwise} \end{cases} \end{aligned}$$

for each $b \in B$.

Proposition 3.3.3 ▶ Properties of Direct Images

Let $f \colon A \to B$ be a function.

1. *Functoriality.* The assignment $U \mapsto f_*(U)$ defines a functor

$$f_* \colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[f_*](U) \stackrel{\text{def}}{=} f_*(U);$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:
  - If $U \subset V$, then $f_*(U) \subset f_*(V)$.

2. *Triple Adjointness.* We have a triple adjunction

$$\left(f_* \dashv f^{-1} \dashv f_!\right)\colon \quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left, "f_*"] \arrow[r, bend right, "f_!"'] & \mathcal{P}(B), \arrow[l, "f^{-1}"'] \end{tikzcd}$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(B)}(f_*(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}\left(U, f^{-1}(V)\right), \\ \mathrm{Hom}_{\mathcal{P}(A)}\left(f^{-1}(U), V\right) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f_!(V)), \end{aligned}$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

(a) The following conditions are equivalent:
   (i) We have $f_*(U) \subset V$;
   (ii) We have $U \subset f^{-1}(V)$;

(b) The following conditions are equivalent:
   (i) We have $f^{-1}(U) \subset V$.
   (ii) We have $U \subset f_!(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$f_*\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} f_*(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f_*(U) \cup f_*(V) &= f_*(U \cup V), \\ f_*(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$f_*\left(\bigcap_{i\in I} U_i\right) \subset \bigcap_{i\in I} f_*(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$\begin{gathered} f_*(U \cap V) \subset f_*(U) \cap f_*(V), \\ f_*(A) \subset B, \end{gathered}$$

natural in $U, V \in \mathcal{P}(A)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(f_*, f_*^{\otimes}, f_{*|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with equalities

$$\begin{gathered} f_{*|U,V}^{\otimes} \colon f_*(U) \cup f_*(V) \xrightarrow{=} f_*(U \cup V), \\ f_{*|\mathbb{1}}^{\otimes} \colon \emptyset \xrightarrow{=} \emptyset, \end{gathered}$$

natural in $U, V \in \mathcal{P}(A)$.

6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure

$$\left(f_*, f_*^{\otimes}, f_{*|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with inclusions

$$\begin{gathered} f_{*|U,V}^{\otimes} \colon f_*(U \cap V) \hookrightarrow f_*(U) \cap f_*(V), \\ f_{*|\mathbb{1}}^{\otimes} \colon f_*(A) \hookrightarrow B, \end{gathered}$$

natural in $U, V \in \mathcal{P}(A)$.

7. *Relation to Direct Images With Compact Support.* We have

$$f_*(U) = B \setminus f_!(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

Proof 3.3.4 ▶ Proof of Proposition 3.3.3

Item 1: Functoriality

Clear.

Item 2: Triple Adjointness

This follows from Kan Extensions, Item 2 of Proposition 1.1.6.

Item 3: Preservation of Colimits

This follows from Item 2 and Categories, ?? of Proposition 6.1.3.

Item 4: Oplax Preservation of Limits

Omitted.

Item 5: Symmetric Strict Monoidality With Respect to Unions

This follows from Item 3.

Item 6: Symmetric Oplax Monoidality With Respect to Intersections

This follows from ??.

Item 7: Relation to Direct Images With Compact Support

Applying Item 7 of Proposition 3.5.5 to $A \setminus U$, we have

$$\begin{aligned} f_!(A \setminus U) &= B \setminus f_*(A \setminus (A \setminus U)) \\ &= B \setminus f_*(U). \end{aligned}$$

Taking complements, we then obtain

$$\begin{aligned} f_*(U) &= B \setminus (B \setminus f_*(U)), \\ &= B \setminus f_!(A \setminus U), \end{aligned}$$

which finishes the proof. ∎

Proposition 3.3.5 ▶ Properties of the Direct Image Function Operation

Let $f: A \longrightarrow B$ be a function.

1. *Functionality I.* The assignment $f \mapsto f_*$ defines a function

$$(-)_*: \mathsf{Sets}(A, B) \longrightarrow \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II.* The assignment $f \mapsto f_*$ defines a function

$$(-)_*: \mathsf{Sets}(A, B) \longrightarrow \mathsf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have
$$(\mathrm{id}_A)_* = \mathrm{id}_{\mathcal{P}(A)};$$
4. *Interaction With Composition.* For each pair of composable functions $f\colon A \to B$ and $g\colon B \to C$, we have
$$(g \circ f)_* = g_* \circ f_*, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "f_*"] \arrow[dr, "(g\circ f)_*"'] & \mathcal{P}(B) \arrow[d, "g_*"] \\ & \mathcal{P}(C). \end{tikzcd}$$

Proof 3.3.6 ▶ Proof of Proposition 3.3.5

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

This follows from Kan Extensions, Item 10 of Proposition 1.1.6.

Item 4: Interaction With Composition

This follows from Kan Extensions, Item 11 of Proposition 1.1.6.

3.4 Inverse Images

Let A and B be sets and let $f\colon A \to B$ be a function.

Definition 3.4.1 ▶ Inverse Images

The **inverse image function associated to** f is the function[1]
$$f^{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$
defined by[2]
$$f^{-1}(V) \overset{\text{def}}{=} \{a \in A \mid \text{we have } f(a) \in V\}$$

for each $V \in \mathcal{P}(B)$.

[1] *Further Notation:* Also written $f^* : \mathcal{P}(B) \to \mathcal{P}(A)$.
[2] *Further Terminology:* The set $f^{-1}(V)$ is called the **inverse image of** V **by** f.

Remark 3.4.2 ▶ Unwinding Definition 3.4.1

Identifying subsets of B with functions from B to $\{\text{true}, \text{false}\}$ via Item 6 of Proposition 3.2.3, we see that the inverse image function associated to f is equivalently the function

$$f^* : \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by

$$f^*(\chi_V) \stackrel{\text{def}}{=} \chi_V \circ f$$

for each $\chi_V \in \mathcal{P}(B)$, where $\chi_V \circ f$ is the composition

$$A \xrightarrow{f} B \xrightarrow{\chi_V} \{\text{true}, \text{false}\}$$

in Sets.

Proposition 3.4.3 ▶ Properties of Inverse Images

Let $f : A \to B$ be a function.

1. *Functoriality.* The assignment $V \mapsto f^{-1}(V)$ defines a functor

$$f^{-1} : (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

where

- *Action on Objects.* For each $V \in \mathcal{P}(B)$, we have

$$[f^{-1}](V) \stackrel{\text{def}}{=} f^{-1}(V);$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(B)$:
  - If $U \subset V$, then $f^{-1}(U) \subset f^{-1}(V)$.

2. *Triple Adjointness.* We have a triple adjunction

$$\left(f_* \dashv f^{-1} \dashv f_!\right): \quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left=50, "f_*"] \arrow[r, bend right=50, "f_!"'] \arrow[r, phantom, bend left=25, "\perp"] \arrow[r, phantom, bend right=25, "\perp"] & \mathcal{P}(B) \arrow[l, "f^{-1}"'] \end{tikzcd},$$

witnessed by bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(B)}(f_*(U), V) \cong \mathrm{Hom}_{\mathcal{P}(A)}\left(U, f^{-1}(V)\right),$$
$$\mathrm{Hom}_{\mathcal{P}(A)}\left(f^{-1}(U), V\right) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f_!(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

(a) The following conditions are equivalent:

(i) We have $f_*(U) \subset V$;

(ii) We have $U \subset f^{-1}(V)$;

(b) The following conditions are equivalent:

(i) We have $f^{-1}(U) \subset V$.

(ii) We have $U \subset f_!(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$f^{-1}\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$f^{-1}(U) \cup f^{-1}(V) = f^{-1}(U \cup V),$$
$$f^{-1}(\emptyset) = \emptyset,$$

natural in $U, V \in \mathcal{P}(B)$.

4. *Preservation of Limits.* We have an equality of sets

$$f^{-1}\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V),$$
$$f^{-1}(B) = A,$$

natural in $U, V \in \mathcal{P}(B)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The inverse image function of Item 1 has a symmetric strict monoidal structure

$$\left(f^{-1}, f^{-1,\otimes}, f_{\not\Vdash}^{-1,\otimes}\right) \colon (\mathcal{P}(B), \cup, \emptyset) \to (\mathcal{P}(A), \cup, \emptyset),$$

being equipped with equalities

$$f_{U,V}^{-1,\otimes} \colon f^{-1}(U) \cup f^{-1}(V) \xrightarrow{=} f^{-1}(U \cup V),$$
$$f_{\not\Vdash}^{-1,\otimes} \colon \emptyset \xrightarrow{=} f^{-1}(\emptyset),$$

natural in $U, V \in \mathcal{P}(B)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The inverse image function of Item 1 has a symmetric strict monoidal structure

$$\left(f^{-1}, f^{-1,\otimes}, f_{\not\Vdash}^{-1,\otimes}\right) \colon (\mathcal{P}(B), \cap, B) \to (\mathcal{P}(A), \cap, A),$$

being equipped with equalities

$$f_{U,V}^{-1,\otimes} \colon f^{-1}(U) \cap f^{-1}(V) \xrightarrow{=} f^{-1}(U \cap V),$$
$$f_{\not\Vdash}^{-1,\otimes} \colon A \xrightarrow{=} f^{-1}(B),$$

natural in $U, V \in \mathcal{P}(B)$.

Proof 3.4.4 ▶ Proof of Proposition 3.4.3

Item 1: Functoriality

Clear.

Item 2: Triple Adjointness

This follows from Kan Extensions, Item 2 of Proposition 1.1.6.

Item 3: Preservation of Colimits

This follows from Item 2 and Categories, **??** of Proposition 6.1.3.

Item 4: Preservation of Limits

This follows from Item 2 and Categories, **??** of Proposition 6.1.3.

Item 5: Symmetric Strict Monoidality With Respect to Unions

This follows from Item 3.

Item 6: Symmetric Strict Monoidality With Respect to Intersections

This follows from Item 4.

Proposition 3.4.5 ▶ Properties of the Inverse Image Function Operation

Let $f\colon A \longrightarrow B$ be a function.

1. *Functionality I.* The assignment $f \mapsto f^{-1}$ defines a function
$$(-)^{-1}\colon \mathsf{Sets}(A, B) \longrightarrow \mathsf{Sets}(\mathcal{P}(B), \mathcal{P}(A)).$$
2. *Functionality II.* The assignment $f \mapsto f^{-1}$ defines a function
$$(-)^{-1}\colon \mathsf{Sets}(A, B) \longrightarrow \mathsf{Pos}((\mathcal{P}(B), \subset), (\mathcal{P}(A), \subset)).$$
3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have
$$\mathrm{id}_A^{-1} = \mathrm{id}_{\mathcal{P}(A)};$$
4. *Interaction With Composition.* For each pair of composable functions $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$, we have
$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}, \qquad \begin{tikzcd} \mathcal{P}(C) \arrow[r, "g^{-1}"] \arrow[dr, "(g\circ f)^{-1}"'] & \mathcal{P}(B) \arrow[d, "f^{-1}"] \\ & \mathcal{P}(A). \end{tikzcd}$$

Proof 3.4.6 ▶ Proof of Proposition 3.4.5

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

This follows from Categories, Item 5 of Proposition 1.4.3.

Item 4: Interaction With Composition

This follows from Categories, Item 2 of Proposition 1.4.3.

3.5 Direct Images With Compact Support

Let A and B be sets and let $f\colon A \to B$ be a function.

Definition 3.5.1 ▶ Direct Images With Compact Support

The **direct image with compact support function associated to** f is the function[1]

$$f_!\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by[2,3]

$$\begin{aligned} f_!(U) &\overset{\text{def}}{=} \left\{ b \in B \,\middle|\, \begin{array}{l} \text{for each } a \in A, \text{ if we have} \\ f(a) = b, \text{ then } a \in U \end{array} \right\} \\ &= \left\{ b \in B \,\middle|\, \text{we have } f^{-1}(b) \subset U \right\} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

[1] *Further Notation:* Also written $\forall_f\colon \mathcal{P}(A) \to \mathcal{P}(B)$. This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

- We have $b \in \forall_f(U)$.
- For each $a \in A$, if $b = f(a)$, then $a \in U$.

[2] *Further Terminology:* The set $f_!(U)$ is called the **direct image with compact support of U by f**.

[3] We also have

$$f_!(U) = B \setminus f_*(A \setminus U);$$

see Item 7 of Proposition 3.5.5.

Remark 3.5.2 ▶ Unwinding Definition 3.5.1

Identifying subsets of A with functions from A to $\{\text{true}, \text{false}\}$ via Item 6 of Proposition 3.2.3, we see that the direct image with compact support function associated to f is equivalently the function

$$f_!\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by

$$f_!(\chi_U) \overset{\text{def}}{=} \mathrm{Ran}_f(\chi_U)$$

$$
\begin{aligned}
&= \lim\left(\left(\underline{(-_1)} \downarrow f\right) \overset{\mathrm{pr}}{\twoheadrightarrow} A \xrightarrow{\chi_U} \{\mathrm{true}, \mathrm{false}\}\right) \\
&= \lim_{\substack{a\in A \\ f(a)=-_1}} (\chi_U(a)) \\
&= \bigwedge_{\substack{a\in A \\ f(a)=-_1}} (\chi_U(a)).
\end{aligned}
$$

So, in other words, we have

$$
\begin{aligned}
[f_!(\chi_U)](b) &= \bigwedge_{\substack{a\in A \\ f(a)=b}} (\chi_U(a)) \\
&= \begin{cases} \mathrm{true} & \text{if, for each } a \in A \text{ such that} \\ & f(a) = b\text{, we have } a \in U, \\ \mathrm{false} & \text{otherwise} \end{cases} \\
&= \begin{cases} \mathrm{true} & \text{if } f^{-1}(b) \subset U \\ \mathrm{false} & \text{otherwise} \end{cases}
\end{aligned}
$$

for each $b \in B$.

Definition 3.5.3 ▶ Image and Complement Parts of Direct Images With Compact Support

Let U be a subset of A.[1,2]

1. The **image part of the direct image with compact support $f_!(U)$ of U** is the set $f_{!,\mathrm{im}}(U)$ defined by

$$
\begin{aligned}
f_{!,\mathrm{im}}(U) &\overset{\mathrm{def}}{=} f_!(U) \cap \mathrm{Im}(f) \\
&= \left\{ b \in B \,\middle|\, \begin{array}{l} \text{we have } f^{-1}(b) \subset U \\ \text{and } f^{-1}(b) \neq \emptyset \end{array} \right\}.
\end{aligned}
$$

2. The **complement part of the direct image with compact support $f_!(U)$ of U** is the set $f_{!,\mathrm{cp}}(U)$ defined by

$$
\begin{aligned}
f_{!,\mathrm{cp}}(U) &\overset{\mathrm{def}}{=} f_!(U) \cap (B \setminus \mathrm{Im}(f)) \\
&= B \setminus \mathrm{Im}(f) \\
&= \left\{ b \in B \,\middle|\, \begin{array}{l} \text{we have } f^{-1}(b) \subset U \\ \text{and } f^{-1}(b) = \emptyset \end{array} \right\}
\end{aligned}
$$

$$= \left\{b \in B \,\middle|\, f^{-1}(b) = \emptyset\right\}.$$

[1] Note that we have

$$f_!(U) = f_{!,\mathrm{im}}(U) \cup f_{!,\mathrm{cp}}(U),$$

as

$$\begin{aligned} f_!(U) &= f_!(U) \cap B \\ &= f_!(U) \cap (\mathrm{Im}(f) \cup (B \setminus \mathrm{Im}(f))) \\ &= (f_!(U) \cap \mathrm{Im}(f)) \cup (f_!(U) \cap (B \setminus \mathrm{Im}(f))) \\ &\overset{\mathrm{def}}{=} f_{!,\mathrm{im}}(U) \cup f_{!,\mathrm{cp}}(U). \end{aligned}$$

[2] In terms of the meet computation of $f_!(U)$ of Remark 3.5.2, namely

$$f_!(\chi_U) = \bigwedge_{\substack{a \in A \\ f(a) = -1}} (\chi_U(a)),$$

we see that $f_{!,\mathrm{im}}$ corresponds to meets indexed over nonempty sets, while $f_{!,\mathrm{cp}}$ corresponds to meets indexed over the empty set.

Example 3.5.4 ▶ Examples of Direct Images With Compact Support

Here are some examples of direct images with compact support.

1. *The Multiplication by Two Map on the Natural Numbers.* Consider the function $f\colon \mathbb{N} \to \mathbb{N}$ given by

$$f(n) \overset{\mathrm{def}}{=} 2n$$

   for each $n \in \mathbb{N}$. Since f is injective, we have

$$\begin{aligned} f_{!,\mathrm{im}}(U) &= f_*(U) \\ f_{!,\mathrm{cp}}(U) &= \{\text{odd natural numbers}\} \end{aligned}$$

   for any $U \subset \mathbb{N}$.

2. *Parabolas.* Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) \overset{\mathrm{def}}{=} x^2$$

   for each $x \in \mathbb{R}$. We have

$$f_{!,\mathrm{cp}}(U) = \mathbb{R}_{<0}$$

   for any $U \subset \mathbb{R}$. Moreover, since $f^{-1}(x) = \left\{-\sqrt{x}, \sqrt{x}\right\}$, we have e.g.:

$$f_{!,\mathrm{im}}([0,1]) = \{0\},$$

$$
\begin{aligned}
f_{!,\mathrm{im}}([-1,1]) &= [0,1],\\
f_{!,\mathrm{im}}([1,2]) &= \emptyset,\\
f_{!,\mathrm{im}}([-2,-1] \cup [1,2]) &= [1,4].
\end{aligned}
$$

3. *Circles.* Consider the function $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by

$$f(x,y) \overset{\mathrm{def}}{=} x^2 + y^2$$

for each $(x,y) \in \mathbb{R}^2$. We have

$$f_{!,\mathrm{cp}}(U) = \mathbb{R}_{<0}$$

for any $U \subset \mathbb{R}^2$, and since

$$f^{-1}(r) = \begin{cases} \text{a circle of radius } r \text{ about the origin} & \text{if } r > 0, \\ \{(0,0)\} & \text{if } r = 0, \\ \emptyset & \text{if } r < 0, \end{cases}$$

we have e.g.:

$$
\begin{aligned}
f_{!,\mathrm{im}}([-1,1] \times [-1,1]) &= [0,1],\\
f_{!,\mathrm{im}}(([-1,1] \times [-1,1]) \setminus [-1,1] \times \{0\}) &= \emptyset.
\end{aligned}
$$

Proposition 3.5.5 ▶ Properties of Direct Images With Compact Support

Let $f\colon A \longrightarrow B$ be a function.

1. *Functoriality.* The assignment $U \mapsto f_!(U)$ defines a functor

$$f_!\colon (\mathcal{P}(A), \subset) \longrightarrow (\mathcal{P}(B), \subset)$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[f_!](U) \overset{\mathrm{def}}{=} f_!(U);$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:
  - If $U \subset V$, then $f_!(U) \subset f_!(V)$.

2. *Triple Adjointness.* We have a triple adjunction

$$\left(f_* \dashv f^{-1} \dashv f_!\right)\colon \quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left=50, "f_*"] \arrow[r, bend right=50, "f_!"'] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathcal{P}(B), \arrow[l, "f^{-1}" description] \end{tikzcd}$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(B)}(f_*(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}\left(U, f^{-1}(V)\right), \\ \mathrm{Hom}_{\mathcal{P}(A)}\left(f^{-1}(U), V\right) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f_!(V)), \end{aligned}$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

   (a) The following conditions are equivalent:
      (i) We have $f_*(U) \subset V$;
      (ii) We have $U \subset f^{-1}(V)$;
   (b) The following conditions are equivalent:
      (i) We have $f^{-1}(U) \subset V$.
      (ii) We have $U \subset f_!(V)$.

3. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i \in I} f_!(U_i) \subset f_!\left(\bigcup_{i \in I} U_i\right),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} f_!(U) \cup f_!(V) &\hookrightarrow f_!(U \cup V), \\ \emptyset &\hookrightarrow f_!(\emptyset), \end{aligned}$$

natural in $U, V \in \mathcal{P}(A)$.

4. *Preservation of Limits.* We have an equality of sets

$$f_!\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} f_!(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$f^{-1}(U \cap V) = f_!(U) \cap f^{-1}(V),$$
$$f_!(A) = B,$$

natural in $U, V \in \mathcal{P}(A)$.

5. *Symmetric Lax Monoidality With Respect to Unions.* The direct image with compact support function of Item 1 has a symmetric lax monoidal structure

$$\left(f_!, f_!^{\otimes}, f_{!|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with inclusions

$$f_{!|U,V}^{\otimes} \colon f_!(U) \cup f_!(V) \hookrightarrow f_!(U \cup V),$$
$$f_{!|\mathbb{1}}^{\otimes} \colon \emptyset \hookrightarrow f_!(\emptyset),$$

natural in $U, V \in \mathcal{P}(A)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(f_!, f_!^{\otimes}, f_{!|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with equalities

$$f_{!|U,V}^{\otimes} \colon f_!(U \cap V) \xrightarrow{=} f_!(U) \cap f_!(V),$$
$$f_{!|\mathbb{1}}^{\otimes} \colon f_!(A) \xrightarrow{=} B,$$

natural in $U, V \in \mathcal{P}(A)$.

7. *Relation to Direct Images.* We have

$$f_!(U) = B \setminus f_*(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

8. *Interaction With Injections.* If f is injective, then we have

$$\begin{aligned} f_{!,\mathrm{im}}(U) &= f_*(U), \\ f_{!,\mathrm{cp}}(U) &= B \setminus \mathrm{Im}(f), \\ f_!(U) &= f_{!,\mathrm{im}}(U) \cup f_{!,\mathrm{cp}}(U) \\ &= f_*(U) \cup (B \setminus \mathrm{Im}(f)) \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

9. *Interaction With Surjections.* If f is surjective, then we have

$$\begin{aligned} f_{!,\mathrm{im}}(U) &\subset f_*(U), \\ f_{!,\mathrm{cp}}(U) &= \emptyset, \\ f_!(U) &\subset f_*(U) \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

PROOF 3.5.6 ▶ PROOF OF PROPOSITION 3.5.5

Item 1: Functoriality

Clear.

Item 2: Triple Adjointness

This follows from Kan Extensions, Item 2 of Proposition 1.1.6.

Item 3: Lax Preservation of Colimits

Omitted.

Item 4: Preservation of Limits

Omitted. This follows from Item 2 and Categories, **??** of Proposition 6.1.3.

Item 5: Symmetric Lax Monoidality With Respect to Unions

This follows from **??**.

Item 6: Symmetric Strict Monoidality With Respect to Intersections

This follows from Item 4.

Item 7: Relation to Direct Images

We claim that $f_!(U) = B \setminus f_*(A \setminus U)$.

- *The First Implication.* We claim that

  $$f_!(U) \subset B \setminus f_*(A \setminus U).$$

  Let $b \in f_!(U)$. We need to show that $b \notin f_*(A \setminus U)$, i.e. that there is no $a \in A \setminus U$ such that $f(a) = b$.

  This is indeed the case, as otherwise we would have $a \in f^{-1}(b)$ and $a \notin U$, contradicting $f^{-1}(b) \subset U$ (which holds since $b \in f_!(U)$).

  Thus $b \in B \setminus f_*(A \setminus U)$.

- *The Second Implication.* We claim that

$$B \setminus f_*(A \setminus U) \subset f_!(U).$$

Let $b \in B \setminus f_*(A \setminus U)$. We need to show that $b \in f_!(U)$, i.e. that $f^{-1}(b) \subset U$.

Since $b \notin f_*(A \setminus U)$, there exists no $a \in A \setminus U$ such that $b = f(a)$, and hence $f^{-1}(b) \subset U$.

Thus $b \in f_!(U)$.

This finishes the proof of Item 7.

Item 8: Interaction With Injections

Clear.

Item 9: Interaction With Surjections

Clear.

Proposition 3.5.7 ▶ Properties of the Direct Image With Compact Support Function Operation

Let $f : A \to B$ be a function.

1. *Functionality I.* The assignment $f \mapsto f_!$ defines a function

$$(-)_! : \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II.* The assignment $f \mapsto f_!$ defines a function

$$(-)_! : \mathsf{Sets}(A, B) \to \mathsf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_A)_! = \mathrm{id}_{\mathcal{P}(A)};$$

4. *Interaction With Composition.* For each pair of composable functions $f : A \to B$ and $g : B \to C$, we have

$$(g \circ f)_! = g_! \circ f_!, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "f_!"] \arrow[dr, "(g \circ f)_!"'] & \mathcal{P}(B) \arrow[d, "g_!"] \\ & \mathcal{P}(C). \end{tikzcd}$$

Proof 3.5.8 ▶ Proof of Proposition 3.5.7

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

This follows from Kan Extensions, Item 10 of Proposition 1.1.6.

Item 4: Interaction With Composition

This follows from Kan Extensions, Item 11 of Proposition 1.1.6.

4 Pointed Sets

4.1 Foundations

Definition 4.1.1 ▶ Pointed Sets

A **pointed set**[1] is equivalently

- An $\mathbb{E}_0$-monoid in $(\mathsf{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$;
- A pointed object in $(\mathsf{Sets}, \mathrm{pt})$.

[1] *Further Terminology:* Also called an $\mathbb{F}_1$**-module**.

Remark 4.1.2 ▶ Unwinding Definition 4.1.1

In detail, a **pointed set** is a pair (X, x_0) consisting of

- *The Underlying Set.* A set X, called the **underlying set of** (X, x_0);
- *The Basepoint.* A morphism

$$[x_0]\colon \mathrm{pt} \to X$$

in Sets, determining an element $x_0 \in X$, called the **basepoint of** X.

Example 4.1.3 ▶ The Zero Sphere

The 0-**sphere**[1] is the pointed set $(S^0, 0)$[2] consisting of

- *The Underlying Set.* The set S^0 defined by

$$S^0 \stackrel{\text{def}}{=} \{0, 1\};$$

- *The Basepoint.* The element 0 of S^0.

[1] *Further Terminology:* Also called the **underlying pointed set of the field with one element**.
[2] *Further Notation:* Also denoted $(\mathbb{F}_1, 0)$.

Example 4.1.4 ▶ The Trivial Pointed Set

The **trivial pointed set** is the pointed set $(\mathrm{pt}, \star)$ consisting of

- *The Underlying Set.* The punctual set $\mathrm{pt} \stackrel{\text{def}}{=} \{\star\}$;
- *The Basepoint.* The element $\star$ of pt.

Example 4.1.5 ▶ The Underlying Pointed Set of a Semimodule

The **underlying pointed set** of a semimodule (M, α_M) is the pointed set $(M, 0_M)$.

Example 4.1.6 ▶ The Underlying Pointed Set of a Module

The **underlying pointed set** of a module (M, α_M) is the pointed set $(M, 0_M)$.

4.2 Morphisms of Pointed Sets

4.2.1 Foundations

Definition 4.2.1 ▶ Morphisms of Pointed Sets

A **morphism of pointed sets**[1] is equivalently

- A morphism of $\mathbb{E}_0$-monoids in $(\mathrm{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$.
- A morphism of pointed objects in $(\mathsf{Sets}, \mathrm{pt})$.

[1] *Further Terminology:* Also called a **pointed function** or a **morphism of $\mathbb{F}_1$-modules**.

Remark 4.2.2 ▶ Unwinding Definition 4.2.1

In detail, a **morphism of pointed sets** $f\colon (X, x_0) \to (Y, y_0)$ is a morphism of sets $f\colon X \to Y$ such that the diagram

$$\begin{tikzcd} & \mathrm{pt} \arrow[dl, "{[x_0]}"'] \arrow[dr, "{[y_0]}"] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd}$$

commutes, i.e. such that

$$f(x_0) = y_0.$$

Definition 4.2.3 ▶ The Category of Pointed Sets

The **category of pointed sets** is the category Sets_* defined equivalently as

- The homotopy category of the ∞-category $\mathsf{Mon}_{\mathbb{E}_0}(\mathrm{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$ of Monoids in Monoidal ∞-Categories, **??**;
- The category Sets_* of Categories, **??**.

Remark 4.2.4 ▶ Unwinding Definition 4.2.3

In detail, the **category of pointed sets** is the category Sets_* where

- *Objects.* The objects of Sets_* are pointed sets;
- *Morphisms.* The morphisms of Sets_* are morphisms of pointed sets;
- *Identities.* For each $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the unit map

  $$\mathbb{1}^{\mathsf{Sets}_*}_{(X,x_0)} \colon \mathrm{pt} \to \mathsf{Sets}_*((X, x_0), (X, x_0))$$

  of Sets_* at (X, x_0) is defined by[1]

  $$\mathrm{id}^{\mathsf{Sets}_*}_{(X,x_0)} \overset{\text{def}}{=} \mathrm{id}_X;$$

- *Composition.* For each $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the composition map

  $$\circ^{\mathsf{Sets}_*}_{(X,x_0),(Y,y_0),(Z,z_0)} \colon \mathsf{Sets}_*((Y, y_0), (Z, z_0)) \times \mathsf{Sets}_*((X, x_0), (Y, y_0)) \to \mathsf{Sets}_*((X, x_0), (Z, z_0))$$

of Sets_* at $((X, x_0), (Y, y_0), (Z, z_0))$ is defined by[2]

$$g \circ^{\mathsf{Sets}_*}_{(X,x_0),(Y,y_0),(Z,z_0)} f \stackrel{\text{def}}{=} g \circ f.$$

[1] Note that id_X is indeed a morphism of pointed sets, as we have $\mathrm{id}_X(x_0) = x_0$.

[2] Note that the composition of two morphisms of pointed sets is indeed a morphism of pointed sets, as we have

$$\begin{aligned} g(f(x_0)) &= g(y_0) \\ &= z_0. \end{aligned}$$

Proposition 4.2.5 ▶ Elementary Properties of Pointed Sets

Let (X, x_0) be a pointed set.

1. *Completeness.* The category Sets_* of pointed sets and morphisms between them is complete:
   (a) *Products.* The product of two pointed sets (X, x_0) and (Y, y_0) is the pair $(X \times Y, (x_0, y_0))$;
   (b) *Pullbacks.* The pullback of two pointed functions

   $$\begin{aligned} f &\colon (X, x_0) \to (Z, z_0), \\ g &\colon (Y, y_0) \to (Z, z_0) \end{aligned}$$

   is the pointed set $\big((X, x_0) \times_{(z,z_0)} (Y, y_0), p_0\big)$ consisting of
   - *The Underlying Set.* The set $(X, x_0) \times_{(z,z_0)} (Y, y_0)$ defined by

   $$(X, x_0) \times_{(z,z_0)} (Y, y_0) \stackrel{\text{def}}{=} \{(x, y) \in X \times Y \mid f(x) = z_0 = g(y)\};$$

   - *The Basepoint.* The element (x_0, y_0) of $(X, x_0) \times_{(z,z_0)} (Y, y_0)$.

   (c) *Equalisers.* The equaliser of a parallel pair of pointed functions

   $$f, g \colon (X, x_0) \to (Y, y_0)$$

   is the pointed set $(\mathrm{Eq}_*(f, g), x_0)$ consisting of
   - *The Underlying Set.* The set $\mathrm{Eq}_*(f, g)$ defined by

   $$\mathrm{Eq}_*(f, g) \stackrel{\text{def}}{=} \{x \in X \mid f(x) = z_0 = g(x)\};$$

   - *The Basepoint.* The element x_0 of $\mathrm{Eq}_*(f, g)$.

2. *Cocompleteness.* The category Sets_* of pointed sets and morphisms between them is cocomplete:

   (a) *Coproducts.* The coproduct of two pointed sets (X, x_0) and (Y, y_0) is their wedge sum $(X \vee Y, p_0)$ of Definition 4.4.1;

   (b) *Pushouts.* The pushout of two pointed functions

   $$\begin{aligned} f\colon (Z, z_0) &\to (X, x_0),\\ g\colon (Z, z_0) &\to (Y, y_0) \end{aligned}$$

   is the pointed set $\left(X \coprod_{f,Z,g} Y, p_0\right)$, where $p_0 = [x_0] = [y_0]$.

   (c) *Coequalisers.* The coequaliser of a parallel pair

   $$f, g\colon (X, x_0) \rightrightarrows (Y, y_0)$$

   of morphisms of pointed sets is given by $(\mathrm{CoEq}(f, g), x_0)$.

3. *Failure To Be Cartesian Closed.* The category Sets_* is not Cartesian closed.

4. *Relation to Partial Functions.* We have an equivalence of categories[1]

   $$\mathsf{Sets}_* \overset{\text{eq.}}{\cong} \mathsf{Sets}^{\text{part.}}$$

   between the category of pointed sets and pointed functions between them and the category of sets and partial functions between them.

[1] *Warning:* This is not an isomorphism of categories, only an equivalence.

Proof 4.2.6 ▶ Proof of Proposition 4.2.5

Item 1: Completeness

Omitted.

Item 2: Cocompleteness

Omitted.

Item 3: Failure To Be Cartesian Closed

See [MSE 2855868].

Item 4: Relation to Partial Functions

Omitted.

4.2.2 Pointed Sets of Morphisms of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 4.2.7 ▶ Pointed Sets of Morphisms of Pointed Sets

The **pointed set of morphisms of pointed sets from** (X, x_0) **to** (Y, y_0) is the pointed set $\mathbf{Sets}_*(X, Y)$ consisting of

- *The Underlying Set.* The set $\mathbf{Sets}_*((X, x_0), (Y, y_0))$ of morphisms of pointed sets from (X, x_0) to (Y, y_0);
- *The Basepoint.* The element

$$\Delta_{y_0} \colon (X, x_0) \longrightarrow (Y, y_0)$$

  of $\mathbf{Sets}_*((X, x_0), (Y, y_0))$.

4.3 Free Pointed Sets

Let X be a set.

Definition 4.3.1 ▶ Free Pointed Sets

The **free pointed set on** X is the pointed set X^+ consisting of

- *The Underlying Set.* The set X^+ defined by

$$X^+ \stackrel{\text{def}}{=} X \coprod \mathrm{pt};$$

- *The Basepoint.* The element $\star$ of X^+.

Proposition 4.3.2 ▶ Properties of Free Pointed Sets

Let X be a set.

1. *Functoriality.* The assignment $X \mapsto X^+$ defines a functor

$$(-)^+ \colon \mathsf{Sets} \longrightarrow \mathsf{Sets}_*,$$

   where

   - *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\left[(-)^+\right](X) \stackrel{\text{def}}{=} X_+,$$

where X_+ is the pointed set of Definition 4.3.1;

- *Action on Morphisms.* For each morphism $f\colon X \to Y$ of Sets, the image

$$f_+\colon X_+ \to Y_+$$

of f by $(-)^+$ is the map of pointed sets defined by

$$f^+(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X, \\ \star & \text{if } x = \star. \end{cases}$$

2. *Adjointness.* We have an adjunction

$$\left((-)^+ \dashv 忘\right)\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "(-)^+"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_* \arrow[l, bend left, "忘"] \end{tikzcd},$$

witnessed by a bijection of sets

$$\mathsf{Sets}_*((X_+, \star), (Y, y_0)) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, y_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

3. *Symmetric Strong Monoidality With Respect to Wedge Sums.* The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^+, (-)^{+,\coprod}, (-)^{+,\coprod}_{\mathbb{1}}\right)\colon (\mathsf{Sets}, \coprod, \emptyset) \to (\mathsf{Sets}_*, \vee, \mathrm{pt}),$$

being equipped with isomorphisms of pointed sets

$$(-)^{+,\coprod}_{X,Y}\colon X^+ \vee Y^+ \xrightarrow{\cong} (X \coprod Y)^+,$$
$$(-)^{+,\coprod}_{\mathbb{1}}\colon \mathrm{pt} \xrightarrow{\cong} \emptyset^+,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Symmetric Strong Monoidality With Respect to Smash Products.* The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^+, (-)^{+,\times}, (-)^{+,\times}_{\mathbb{1}}\right)\colon (\mathsf{Sets}, \times, \mathrm{pt}) \to \left(\mathsf{Sets}_*, \wedge, S^0\right),$$

being equipped with isomorphisms of pointed sets

$$(-)^{+,\times}_{X,Y}\colon X^+ \wedge Y^+ \xrightarrow{\cong} (X \times Y)^+,$$
$$(-)^{+,\times}_{\mathbb{1}}\colon S^0 \xrightarrow{\cong} \mathrm{pt}^+,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

Proof 4.3.3 ▶ Proof of Proposition 4.3.2

Item 1: Functoriality

Clear.

Item 2: Adjointness

Clear.

Item 3: Symmetric Strong Monoidality With Respect to Wedge Sums

Omitted.

Item 4: Symmetric Strong Monoidality With Respect to Smash Products

Omitted.

4.4 Wedge Sums of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 4.4.1 ▶ Wedge Sums of Pointed Sets

The **wedge sum of X and Y** is the pointed set $(X \vee Y, p_0)$ consisting of

- *The Underlying Set.* The set $X \vee Y$ defined by[1]

$$\begin{aligned} (X \vee Y, p_0) &\stackrel{\text{def}}{=} (X, x_0) \coprod (Y, y_0) \\ &\cong \left(X \coprod\nolimits_{\mathrm{pt}} Y, p_0\right) \\ &\cong \left(X \coprod Y/\sim, p_0\right), \end{aligned} \qquad \begin{tikzcd} X \vee Y \arrow[dr, phantom, very near start, "\ulcorner"] & Y \arrow[l] \\ X \arrow[u] & \mathrm{pt}, \arrow[l, hook', "{[x_0]}"] \arrow[u, hook, "{[y_0]}"'] \end{tikzcd}$$

  where $\sim$ is the equivalence relation on $X \coprod Y$ given by $x_0 \sim y_0$;

- *The Basepoint.* The element p_0 of $X \vee Y$ defined by

$$\begin{aligned} p_0 &\stackrel{\text{def}}{=} [x_0] \\ &= [y_0]. \end{aligned}$$

[1] Here $(X, x_0) \coprod (Y, y_0)$ is the coproduct of (X, x_0) and (Y, y_0) in Sets_*.

Proposition 4.4.2 ▶ Properties of Wedge Sums of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments $(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \vee Y, p_0)$ define functors

$$\begin{aligned} X \vee -\colon \mathsf{Sets}_* &\longrightarrow \mathsf{Sets}_*, \\ - \vee Y\colon \mathsf{Sets}_* &\longrightarrow \mathsf{Sets}_*, \\ -_1 \vee -_2\colon \mathsf{Sets}_* \times \mathsf{Sets}_* &\longrightarrow \mathsf{Sets}_*. \end{aligned}$$

2. *Associativity.* We have an isomorphism of pointed sets

$$(X \vee Y) \vee Z \cong X \vee (Y \vee Z),$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathsf{Sets}_*$.

3. *Unitality.* We have isomorphisms of pointed sets

$$\begin{aligned} \mathrm{pt} \vee X &\cong X, \\ X \vee \mathrm{pt} &\cong X, \end{aligned}$$

natural in $(X, x_0) \in \mathsf{Sets}_*$.

4. *Commutativity.* We have an isomorphism of pointed sets

$$X \vee Y \cong Y \vee X,$$

natural in $(X, x_0), (Y, y_0) \in \mathsf{Sets}_*$.

5. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \vee, \mathrm{pt})$ is a symmetric monoidal category.

6. *Symmetric Strong Monoidality With Respect to Free Pointed Sets.* The free pointed set functor of Item 1 of Proposition 4.3.2 has a symmetric strong monoidal structure

$$\left((-)^+, (-)^{+,\coprod}, (-)^{+,\coprod}_{\mathbb{1}}\right)\colon (\mathsf{Sets}, \coprod, \emptyset) \longrightarrow (\mathsf{Sets}_*, \vee, \mathrm{pt}),$$

being equipped with isomorphisms of pointed sets

$$\begin{aligned} (-)^{+,\coprod}_{X,Y}\colon X^+ \vee Y^+ &\xrightarrow{\cong} (X \coprod Y)^+, \\ (-)^{+,\coprod}_{\mathbb{1}}\colon \mathrm{pt} &\xrightarrow{\cong} \emptyset^+, \end{aligned}$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

7. *The Fold Map*. We have a natural transformation

$$\nabla\colon \vee \circ \Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*} \Longrightarrow \mathrm{id}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd} & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[dr, "\vee"] \arrow[d, phantom, "\Downarrow\nabla"] & \\ \mathsf{Sets}_* \arrow[ur, "\Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*}"] \arrow[rr, bend right, "\mathrm{id}_{\mathsf{Sets}_*}"'] & & \mathsf{Sets}_*, \end{tikzcd}$$

called the **fold map**, whose component

$$\nabla_X\colon X \vee X \to X$$

at X is given by the composition

$$\begin{aligned} X &\xrightarrow{\Delta_X} X \times X \\ &\twoheadrightarrow X \times X/{\sim} \\ &\overset{\text{def}}{=} X \vee X. \end{aligned}$$

Proof 4.4.3 ▶ Proof of Proposition 4.4.2

Item 1: Functoriality

Omitted.

Item 2: Associativity

Omitted.

Item 3: Unitality

Omitted.

Item 4: Commutativity

Omitted.

Item 5: Symmetric Monoidality

Omitted.

Item 6: Symmetric Strong Monoidality With Respect to Free Pointed Sets

Omitted.

Item 7: The Fold Map

Omitted.

4.5 Bilinear Morphisms of Pointed Sets

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

Definition 4.5.1 ▶ Bilinear Morphisms of Pointed Sets

A **bilinear morphism of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is a map of sets

$$f\colon (X \times Y, (x_0, y_0)) \to (Z, z_0)$$

satisfying the following conditions:[1,2]

1. *Left Unital Bilinearity.* The diagram

$$\begin{tikzcd}
 & \mathrm{pt} \times \mathrm{pt} \arrow[dr, dashed, "\sim"] & \\
\mathrm{pt} \times Y \arrow[ur, "\mathrm{id}_{\mathrm{pt}} \times \epsilon_Y"] \arrow[d, "{[x_0]} \times \mathrm{id}_Y"'] & & \mathrm{pt} \arrow[d, "{[z_0]}"] \\
X \times Y \arrow[rr, "f"'] & & Z
\end{tikzcd}$$

   commutes, i.e. for each $y \in Y$, we have

$$f(x_0, y) = z_0.$$

2. *Right Unital Bilinearity.* The diagram

$$\begin{tikzcd}
 & \mathrm{pt} \times \mathrm{pt} \arrow[dr, dashed, "\sim"] & \\
X \times \mathrm{pt} \arrow[ur, "\epsilon_X \times \mathrm{id}_{\mathrm{pt}}"] \arrow[d, "\mathrm{id}_X \times {[y_0]}"'] & & \mathrm{pt} \arrow[d, "{[z_0]}"] \\
X \times Y \arrow[rr, "f"'] & & Z
\end{tikzcd}$$

   commutes, i.e. for each $x \in X$, we have

$$f(x, y_0) = z_0.$$

[1]That is, f preserves basepoints on each argument.
[2]Succinctly, f is bilinear if, for each $x \in X$ and each $y \in Y$, we have

$$f(x_0, y) = z_0,$$
$$f(x, y_0) = z_0.$$

Definition 4.5.2 ▶ The Set of Bilinear Morphisms of Pointed Sets

The **set of bilinear morphisms of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is the set $\mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z)$ defined by

$$\mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z) \overset{\text{def}}{=} \{f \in \mathsf{Sets}_*(A \times B, C) \mid f \text{ is bilinear}\}.$$

4.6 Smash Products of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 4.6.1 ▶ Smash Products of Pointed Sets

The **smash product of** (X, x_0) **and** (Y, y_0)[1] is the pointed set $X \wedge Y$[2] such that we have a bijection

$$\mathsf{Sets}_*(X \wedge Y, Z) \cong \mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z),$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

[1]*Further Terminology:* Also called the **tensor product of** $\mathbb{F}_1$**-modules of** (X, x_0) **and** (Y, y_0) or the **tensor product of** (X, x_0) **and** (Y, y_0) **over** $\mathbb{F}_1$.
[2]*Further Notation:* Also written $X \otimes_{\mathbb{F}_1} Y$.

Remark 4.6.2 ▶ Unwinding Definition 4.6.1

In detail, the **smash product of** (X, x_0) **and** (Y, y_0) is the pair $((X \wedge Y, [(x_0, y_0)], \iota)$ consisting of

- A pointed set $X \wedge Y$;
- A bilinear morphism of pointed sets $\iota\colon (X \times Y, (x_0, y_0)) \longrightarrow X \wedge Y$;

satisfying the following universal property:

(UP) Given another such pair $((Z, z_0), f)$ consisting of

- A pointed set (Z, z_0);

- A bilinear morphism of pointed sets $f\colon (X \times Y, (x_0, y_0)) \to X \wedge Y$;

there exists a unique morphism of pointed sets $X \wedge Y \xrightarrow{\exists!} Z$ making the diagram

$$\begin{tikzcd} & X \wedge Y \arrow[d, dashed, "\exists!"] \\ X \times Y \arrow[ur, "\iota"] \arrow[r, "f"'] & Z \end{tikzcd}$$

commute.

Construction 4.6.3 ▶ Smash Products of Pointed Sets

Concretely, the **smash product of** (X, x_0) **and** (Y, y_0) is the pointed set $(X \wedge Y, [(x_0, y_0)])$ consisting of[1]

- *The Underlying Set.* The set $X \wedge Y$ defined by

$$\begin{aligned} X \wedge Y &\cong \mathrm{pt} \coprod_{X \vee Y} (X \times Y) \\ &\overset{\text{def}}{=} \frac{X \times Y}{X \vee Y} \\ &\cong X \times Y/\sim, \end{aligned} \qquad \begin{tikzcd} X \wedge Y \arrow[dr, phantom, "\ulcorner"] & X \times Y \arrow[l] \\ \mathrm{pt} \arrow[u] & X \vee Y, \arrow[l, "!"] \arrow[u, hook] \end{tikzcd}$$

where $\sim$ is the equivalence relation of $X \times Y$ obtained by declaring $(x, y) \sim (x', y')$ iff $(x, y), (x', y') \in X \vee Y$, i.e. by declaring

$$\begin{aligned} (x_0, y) &\sim (x_0, y'), \\ (x, y_0) &\sim (x', y_0) \end{aligned}$$

for all $x \in X$ and all $y \in Y$;

- *The Basepoint.* The element $[(x_0, y_0)]$ of $X \wedge Y$ given by the equivalence class of (x_0, y_0) under the equivalence relation $\sim$ on $X \times Y$.

[1] *Further Notation:* We write $x \wedge y$ for the image of (x, y) under the quotient map

$$X \times Y \twoheadrightarrow \underbrace{\frac{X \times Y}{X \vee Y}}_{\overset{\text{def}}{=} X \wedge Y}$$

Note that we have

$$\begin{aligned} x \wedge y_0 &= x' \wedge y_0, \\ x_0 \wedge y &= x_0 \wedge y' \end{aligned}$$

for each $x, x' \in X$ and each $y, y' \in Y$.

Proof 4.6.4 ▶ Proof of Construction 4.6.3

Clear. ∎

Example 4.6.5 ▶ Examples of Smash Products of Pointed Sets

Here are some examples of smash products of pointed sets.

1. *Smashing With S^0.* For any pointed set X, we have isomorphisms of pointed sets

$$\begin{aligned} S^0 \wedge X &\cong X, \\ X \wedge S^0 &\cong X. \end{aligned}$$

Proposition 4.6.6 ▶ Properties of Smash Products of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments $(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto X \wedge Y$ define functors

$$\begin{aligned} X \wedge -\colon \mathsf{Sets}_* &\longrightarrow \mathsf{Sets}_*, \\ - \wedge Y\colon \mathsf{Sets}_* &\longrightarrow \mathsf{Sets}_*, \\ -_1 \wedge -_2\colon \mathsf{Sets}_* \times \mathsf{Sets}_* &\longrightarrow \mathsf{Sets}_*. \end{aligned}$$

2. *Adjointness.* We have adjunctions

$$(X \wedge - \dashv \mathbf{Sets}_*(X, -))\colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "X \wedge -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(X,-)"] \end{tikzcd}$$

$$(- \wedge Y \dashv \mathbf{Sets}_*(Y, -))\colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "- \wedge Y"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(Y,-)"] \end{tikzcd}$$

witnessed by bijections

$$\begin{aligned} \mathsf{Sets}_*(X \wedge Y, Z) &\cong \mathsf{Sets}_*(X, \mathbf{Sets}_*(Y, Z)), \\ \mathsf{Sets}_*(X \wedge Y, Z) &\cong \mathsf{Sets}_*(X, \mathbf{Sets}_*(A, Z)), \end{aligned}$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, which internalise to isomorphisms of pointed sets

$$\mathbf{Sets}_*(X \wedge Y, Z) \cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(Y, Z)),$$

$$\mathbf{Sets}_*(X \wedge Y, Z) \cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(A, Z)),$$

again natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

3. *Closed Symmetric Monoidality.* The quadruple $\left(\mathsf{Sets}_*, \wedge, S^0, \mathbf{Sets}_*\right)$ is a closed symmetric monoidal category.

4. *Morphisms From the Monoidal Unit.* We have a bijection of sets[1]

$$\mathsf{Sets}_*\left(S^0, X\right) \cong X,$$

natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, internalising also to an isomorphism of pointed sets

$$\mathbf{Sets}_*\left(S^0, X\right) \cong (X, x_0),$$

again natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

5. *Symmetric Strong Monoidality With Respect to Free Pointed Sets.* The free pointed set functor of Item 1 of Proposition 4.3.2 has a symmetric strong monoidal structure

$$\left((-)^+, (-)^{+,\times}, (-)^{+,\times}_{\mathbb{1}}\right) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \to \left(\mathsf{Sets}_*, \wedge, S^0\right),$$

being equipped with isomorphisms of pointed sets

$$(-)^{+,\times}_{X,Y} \colon X^+ \wedge Y^+ \xrightarrow{\cong} (X \times Y)^+,$$
$$(-)^{+,\times}_{\mathbb{1}} \colon S^0 \xrightarrow{\cong} \mathrm{pt}^+,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

6. *Distributivity Over Wedge Sums.* We have isomorphisms of pointed sets

$$X \wedge (Y \vee Z) \cong (X \wedge Y) \vee (X \wedge Z),$$
$$(X \vee Y) \wedge Z \cong (X \wedge Z) \vee (Y \wedge Z),$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

7. *Universal Property I.* The symmetric monoidal structure on the category Sets_* is uniquely determined by the following requirements:

(a) *Two-Sided Preservation of Colimits.* The smash product

$$\wedge \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

of Sets_* preserves colimits separately in each variable.

(b) *The Unit Object Is S^0.* We have $\mathbb{1}_{\mathsf{Sets}_*} = S^0$.

8. *Universal Property II.* The symmetric monoidal structure on the category Sets_* is the unique symmetric monoidal structure on Sets_* such that the free pointed set functor

$$(-)^+ \colon \mathsf{Sets} \to \mathsf{Sets}_*$$

admits a symmetric monoidal structure.

9. *Existence of Monoidal Diagonals.* The triple $(\mathsf{Sets}_*, \wedge, S^0)$ is a monoidal category with diagonals:

(a) *Monoidal Diagonals.* The natural transformation

$$\Delta \colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \wedge \circ \boldsymbol{\Delta}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd} \mathsf{Sets}_* \arrow[rr, bend left=10, "\mathrm{id}_{\mathsf{Sets}_*}"] \arrow[dr, "\boldsymbol{\Delta}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \arrow[d, phantom, "\Downarrow \Delta"] & \mathsf{Sets}_* \\ & \mathsf{Sets}_* \times \mathsf{Sets}_*, \arrow[ur, "\wedge"'] & \end{tikzcd}$$

whose component

$$\Delta_X \colon (X, x_0) \to (X \wedge X, [(x_0, x_0)])$$

at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by the composition

$$\begin{aligned} (X, x_0) &\xrightarrow{\Delta_X} (X \times X, (x_0, x_0)) \\ &\twoheadrightarrow \left(\tfrac{X \times X}{X \vee X}, [(x_0, x_0)]\right) \\ &\stackrel{\text{def}}{=} (X \wedge X, [(x_0, x_0)]) \end{aligned}$$

in Sets_*, is a monoidal natural transformation:

(i) *Naturality*. For each morphism $f\colon X \to Y$ of pointed sets, the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\Delta_X"'] & Y \arrow[d, "\Delta_Y"] \\ X \wedge X \arrow[r, "f \wedge f"'] & Y \wedge Y \end{tikzcd}$$

commutes.

(ii) *Compatibility With Strong Monoidality Constraints*. For each $(X, x_0), (Y, y_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the diagram

$$\begin{tikzcd} X \wedge Y \arrow[r, "\Delta_X \wedge \Delta_Y"] \arrow[d, equal] & (X \wedge X) \wedge (Y \wedge Y) \arrow[d, dashed, "\wr"] \\ X \wedge Y \arrow[r, "\Delta_{X \wedge Y}"'] & (X \wedge Y) \wedge (X \wedge Y) \end{tikzcd}$$

commutes.

(iii) *Compatibility With Strong Unitality Constraints*. The diagram

$$\begin{tikzcd} S^0 \arrow[d, equal] \arrow[dr, "\left(\lambda^{\mathsf{Sets}_*}_{S^0}\right)^{-1} = \left(\rho^{\mathsf{Sets}_*}_{S^0}\right)^{-1}"] & \\ S^0 \arrow[r, "\Delta_{S^0}"'] & S^0 \wedge S^0 \end{tikzcd}$$

commutes.

(b) *The Diagonal of the Unit*. The component

$$\Delta^{\mathsf{Sets}_*}_{S^0}\colon S^0 \xrightarrow{\cong} S^0 \wedge S^0$$

of the diagonal natural transformation of Sets_* at S^0 is an isomorphism.

10. *Comonoids in* Sets_*. The symmetric monoidal functor

$$\left((-)^+, (-)^{+,\times}, (-)^{+,\times}_{\mathbb{1}}\right)\colon (\mathsf{Sets}, \times, \mathrm{pt}) \to \left(\mathsf{Sets}_*, \wedge, S^0\right),$$

of Item 4 of Proposition 4.3.2 lifts to an equivalence of categories

$$\begin{aligned}\mathsf{CoMon}\left(\mathsf{Sets}_*, \wedge, S^0\right) &\overset{\text{eq.}}{\cong} \mathsf{CoMon}(\mathsf{Sets}, \times, \mathrm{pt})\\ &\cong \mathsf{Sets}.\end{aligned}$$

[1] In other words, the forgetful functor

$$忘\colon \mathsf{Sets}_* \to \mathsf{Sets}$$

defined on objects by sending a pointed set to its underlying set is corepresentable by S^0.

Proof 4.6.7 ▶ Proof of Proposition 4.6.6

Item 1: Functoriality

Omitted.

Item 2: Adjointness

Omitted.

Item 3: Closed Symmetric Monoidality

Omitted.

Item 4: Morphisms From the Monoidal Unit

Omitted.

Item 5: Symmetric Strong Monoidality With Respect to Free Pointed Sets

Omitted.

Item 6: Distributivity Over Wedge Sums

This follows from Item 3, Monoidal Categories, Item 7 of Proposition 8.2.13, and the fact that $\vee$ is the coproduct in Sets_*.

Item 7: Universal Property I

Omitted.

Item 8: Universal Property II

See [GGN15, Theorem 5.1].

Item 9: Existence of Monoidal Diagonals

Omitted.

Item 10: Comonoids in Sets_*

See [PS19, Lemma 2.4]. □

4.7 Tensors and Cotensors by Sets

Definition 4.7.1 ▶ Tensors and Cotensors of Pointed Sets by Sets

Let (X, x_0) be a pointed set and let A be a set.

- The **tensor of** (X, x_0) **by** A is the pointed set $A \odot (X, x_0)$ satisfying the following universal property:

  (**UP**) We have a bijection

  $$\mathsf{Sets}_*(A \odot X, K) \cong \mathsf{Sets}(A, \mathbf{Sets}_*(X, K)),$$

  natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

- The **cotensor of** (X, x_0) **by** A is the pointed set $A \pitchfork (X, x_0)$ satisfying the following universal property:

  (**UP**) We have a bijection

  $$\mathsf{Sets}_*(K, A \pitchfork X) \cong \mathsf{Sets}(A, \mathbf{Sets}_*(K, X)),$$

  natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

Remark 4.7.2 ▶ Unwinding Definition 4.7.1, I: Explicit Construction

Let (X, x_0) be a pointed set and let A be a set.

- In detail, the **tensor of** (X, x_0) **by** A is the pointed set $A \odot (X, x_0)$ consisting of
  - *The Underlying Set.* The set $A \odot X$ given by

    $$A \odot X \cong \bigvee_{a \in A} (X, x_0);$$

  - *The Basepoint.* The point $[x_0]$ of $\bigvee_{a \in A}(X, x_0)$.
- In detail, the **cotensor of** (X, x_0) **by** A is the pointed set $A \pitchfork (X, x_0)$ consisting of
  - *The Underlying Set.* The set $A \pitchfork X$ given by

    $$A \pitchfork X \cong \bigwedge_{a \in A} (X, x_0);$$

  - *The Basepoint.* The point $[(x_0, x_0, x_0, \ldots)]$ of $\bigwedge_{a \in A}(X, x_0)$.

Remark 4.7.3 ▶ Unwinding Definition 4.7.1, II: Universal Property

Let A be a set and let (X, x_0) and (K, k_0) be pointed sets.

- The tensor of (X, x_0) by A satisfies the following universal property:

$$\begin{aligned}\mathsf{Sets}_*(A \odot X, K) &\cong \mathsf{Bil}_{\mathbb{E}_0}(A \times X, K)\\ &\overset{\text{def}}{=} \left\{ f \in \mathsf{Sets}(A \times X, K) \,\middle|\, \begin{array}{l}\text{for each } a \in A, \text{ we}\\ \text{have } f(a, x_0) = k_0\end{array} \right\}.\end{aligned}$$

- The cotensor of (X, x_0) by A satisfies the following universal property:

$$\begin{aligned}\mathsf{Sets}_*(K, A \pitchfork X) &\cong \mathsf{Bil}_{\mathbb{E}_0}(A \times K, X)\\ &\overset{\text{def}}{=} \left\{ f \in \mathsf{Sets}(A \times K, X) \,\middle|\, \begin{array}{l}\text{for each } a \in A, \text{ we}\\ \text{have } f(a, k_0) = x_0\end{array} \right\}.\end{aligned}$$

4.8 The Left and Right Tensor Products of Pointed Sets

Definition 4.8.1 ▶ The Left and Right Tensor Products of Pointed Sets

Let (X, x_0) be a pointed set.

- The **left tensor product of pointed sets** is the functor

$$\lhd_{\mathsf{Sets}_*} \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*$$

  defined as the composition

$$\mathsf{Sets}_* \times \mathsf{Sets}_* \xrightarrow{\mathrm{id} \times 忘} \mathsf{Sets}_* \times \mathsf{Sets} \xrightarrow{\beta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*, \mathsf{Sets}}} \mathsf{Sets} \times \mathsf{Sets}_* \xrightarrow{\odot} \mathsf{Sets}_*.$$

- The **right tensor product of pointed sets** is the functor

$$\rhd_{\mathsf{Sets}_*} \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*$$

  defined as the composition

$$\mathsf{Sets}_* \times \mathsf{Sets}_* \xrightarrow{忘 \times \mathrm{id}} \mathsf{Sets} \times \mathsf{Sets}_* \xrightarrow{\odot} \mathsf{Sets}_*.$$

Remark 4.8.2 ▶ Unwinding Definition 4.8.1, I: Explicit Description

Let (X, x_0) and (Y, y_0) be pointed sets.

- In detail, the **left tensor product of** (X, x_0) **and** (Y, y_0) is the pointed set $(X \lhd_{\mathsf{Sets}_*} Y, [x_0])$ consisting of[1]
  - *The Underlying Set.* The set $X \lhd_{\mathsf{Sets}_*} Y$ defined by
  $$\begin{aligned} X \lhd_{\mathsf{Sets}_*} Y &\overset{\text{def}}{=} |Y| \odot X \\ &\cong \bigvee_{y \in Y} (X, x_0); \end{aligned}$$
  - *The Underlying Basepoint.* The point $[x_0]$ of $\bigvee_{y \in Y} (X, x_0)$.
- In detail, the **right tensor product of** (X, x_0) **and** (Y, y_0) is the pointed set $(X \rhd_{\mathsf{Sets}_*} Y, [y_0])$ consisting of[2]
  - *The Underlying Set.* The set $X \rhd_{\mathsf{Sets}_*} Y$ defined by
  $$\begin{aligned} X \rhd_{\mathsf{Sets}_*} Y &\overset{\text{def}}{=} |X| \odot Y \\ &\cong \bigvee_{x \in X} (Y, y_0); \end{aligned}$$
  - *The Underlying Basepoint.* The point $[y_0]$ of $\bigvee_{x \in X} (Y, y_0)$.

[1] *Further Notation:* We write $x \lhd_{\mathsf{Sets}_*} y$ for the image of (x, y) under the map

$$X \times Y \to \underbrace{X \lhd_{\mathsf{Sets}_*} Y}_{\cong \bigvee_{y \in Y} (X, x_0)}.$$

sending (x, y) to the element $x \in X$ in the yth copy of X in $\bigvee_{y \in Y} (X, x_0)$. Note that we have

$$x_0 \lhd_{\mathsf{Sets}_*} y = x_0 \lhd_{\mathsf{Sets}_*} y',$$

for each $y, y' \in Y$.

[2] *Further Notation:* We write $x \rhd_{\mathsf{Sets}_*} y$ for the image of (x, y) under the map

$$X \times Y \to \underbrace{X \rhd_{\mathsf{Sets}_*} Y}_{\cong \bigvee_{x \in X} (Y, y_0)}.$$

sending (x, y) to the element $y \in Y$ in the xth copy of Y in $\bigvee_{x \in X} (Y, y_0)$. Note that we have

$$x \rhd_{\mathsf{Sets}_*} y_0 = x' \rhd_{\mathsf{Sets}_*} y_0,$$

for each $x, x' \in X$.

Remark 4.8.3 ▶ Unwinding Definition 4.8.1, II: Universal Property

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

- The left tensor product of pointed sets satisfies the following universal property:

$$\mathsf{Sets}_*(X \lhd_{\mathsf{Sets}_*} Y, Z) \cong \left\{ f \in \mathsf{Sets}(X \times Y, Z) \,\middle|\, \begin{array}{l}\text{for each } y \in Y, \text{ we}\\ \text{have } f(x_0, y) = z_0\end{array} \right\}.$$

- The right tensor product of pointed sets satisfies the following universal property:

$$\mathsf{Sets}_*(X \rhd_{\mathsf{Sets}_*} Y, Z) \cong \left\{ f \in \mathsf{Sets}(X \times Y, Z) \,\middle|\, \begin{array}{l}\text{for each } x \in X, \text{ we}\\ \text{have } f(x, y_0) = z_0\end{array} \right\}.$$

Proposition 4.8.4 ▶ Properties of the Left and Right Tensor Products of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto X \lhd_{\mathsf{Sets}_*} Y, X \rhd_{\mathsf{Sets}_*} Y$ define functors

$$\begin{aligned} X \lhd_{\mathsf{Sets}_*} -&\colon \mathsf{Sets}_* \to \mathsf{Sets}_*,\\ - \lhd_{\mathsf{Sets}_*} Y&\colon \mathsf{Sets}_* \to \mathsf{Sets}_*,\\ -_1 \lhd_{\mathsf{Sets}_*} -_2&\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*, \end{aligned}$$

   and functors

$$\begin{aligned} X \rhd_{\mathsf{Sets}_*} -&\colon \mathsf{Sets}_* \to \mathsf{Sets}_*,\\ - \rhd_{\mathsf{Sets}_*} Y&\colon \mathsf{Sets}_* \to \mathsf{Sets}_*,\\ -_1 \rhd_{\mathsf{Sets}_*} -_2&\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*. \end{aligned}$$

2. *Skew Monoidality.* The left and right tensor products of pointed sets are part of the structure of a left/right skew monoidal structure on Sets_*:

   - The sextuple $\left(\mathsf{Sets}_*, \lhd_{\mathsf{Sets}_*}, S^0, \alpha^{\mathsf{Sets}_*,\lhd}, \lambda^{\mathsf{Sets}_*,\lhd}, \rho^{\mathsf{Sets}_*,\lhd}\right)$ consisting of

     - *The Underlying Category.* The category Sets_* of pointed sets;

- *The Skew Monoidal Product.* The left tensor product functor

$$\lhd_{\mathsf{Sets}_*} \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*,$$

of Item 1;
- *The Skew Monoidal Unit.* The functor

$$\mathbb{1}^{\mathsf{Sets}_*} \colon \mathrm{pt} \longrightarrow \mathsf{Sets}_*$$

defined by

$$\mathbb{1}_{\mathsf{Sets}_*} \overset{\text{def}}{=} S^0;$$

- *The Skew Associators.* The natural isomorphism

$$\alpha^{\mathsf{Sets}_*,\lhd} \colon \lhd_{\mathsf{Sets}_*} \circ \left(\lhd_{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}\right) \overset{\cong}{\Longrightarrow} \lhd_{\mathsf{Sets}_*} \circ \left(\mathrm{id}_{\mathsf{Sets}_*} \times \lhd_{\mathsf{Sets}_*}\right),$$

whose component

$$\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z} \colon \left(X \lhd_{\mathsf{Sets}_*} Y\right) \lhd_{\mathsf{Sets}_*} Z \xrightarrow{\cong} X \lhd_{\mathsf{Sets}_*} \left(Y \lhd_{\mathsf{Sets}_*} Z\right)$$

at (X,Y,Z) is given by the composition

$$\begin{aligned}
\left(X \lhd_{\mathsf{Sets}_*} Y\right) \lhd_{\mathsf{Sets}_*} Z &\overset{\text{def}}{=} |Z| \odot \left(X \lhd_{\mathsf{Sets}_*} Y\right)\\
&\overset{\text{def}}{=} |Z| \odot (|Y| \odot X)\\
&\cong \bigvee_{z\in Z} (|Y| \odot X, [x_0])\\
&\overset{\text{def}}{=} \bigvee_{z\in Z}\left(\bigvee_{y\in Y}(X,x_0)\right)\\
&\cong \bigvee_{(z,y)\in\bigvee_{z\in Z}(Y,y_0)} (X,x_0)\\
&\overset{\text{def}}{=} \bigvee_{(z,y)\in|Z|\odot Y} (X,x_0)\\
&\cong ||Z| \odot Y| \odot X\\
&\overset{\text{def}}{=} |Y \lhd_{\mathsf{Sets}_*} Z| \odot X\\
&\overset{\text{def}}{=} X \lhd_{\mathsf{Sets}_*} \left(Y \lhd_{\mathsf{Sets}_*} Z\right),
\end{aligned}$$

where the isomorphism

$$\bigvee_{z\in Z}\left(\bigvee_{y\in Y}(X,x_0)\right) \cong \bigvee_{(y,z)\in\bigvee_{z\in Z}(Y,y_0)} (X,x_0)$$

is given by $[(z,(y,x))] \mapsto [((z,y),x)]$. In other words, $\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}$ acts on elements as

$$\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}\big((x \lhd_{\mathsf{Sets}_*} y) \lhd_{\mathsf{Sets}_*} z\big) \stackrel{\text{def}}{=} x \lhd_{\mathsf{Sets}_*} (y \lhd_{\mathsf{Sets}_*} z)$$

for each $(x \lhd_{\mathsf{Sets}_*} y) \lhd_{\mathsf{Sets}_*} z \in (X \lhd_{\mathsf{Sets}_*} Y) \lhd_{\mathsf{Sets}_*} Z$;

- *The Skew Left Unitors.* The natural transformation

$$\lambda^{\mathsf{Sets}_*,\lhd} \colon \lhd_{\mathsf{Sets}_*} \circ \Big(\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}\Big) \Longrightarrow \mathrm{id}_{\mathsf{Sets}_*},$$

whose component

$$\lambda^{\mathsf{Sets}_*,\lhd}_X \colon S^0 \lhd_{\mathsf{Sets}_*} X \to X$$

at X is given by the composition

$$\begin{aligned} S^0 \lhd_{\mathsf{Sets}_*} X &\cong |X| \odot S^0 \\ &\cong \bigvee_{x\in X} S^0 \\ &\to X \end{aligned}$$

where $\bigvee_{x\in X} S^0 \to X$ is the map given by

$$\begin{aligned} (x,0) &\mapsto x, \\ (x,1) &\mapsto x. \end{aligned}$$

In other words, $\lambda^{\mathsf{Sets}_*,\lhd}_X$ acts on elements as

$$\begin{aligned} \lambda^{\mathsf{Sets}_*,\lhd}_X(x \lhd_{\mathsf{Sets}_*} 0) &\stackrel{\text{def}}{=} x, \\ \lambda^{\mathsf{Sets}_*,\lhd}_X(x \lhd_{\mathsf{Sets}_*} 1) &\stackrel{\text{def}}{=} x, \end{aligned}$$

for each $x \in X$;

- *The Skew Right Unitors.* The natural transformation

$$\rho^{\mathsf{Sets}_*,\lhd} \colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \lhd_{\mathsf{Sets}_*} \circ \Big(\mathrm{id}_{\mathsf{Sets}_*} \times \mathbb{1}^{\mathsf{Sets}_*}\Big),$$

whose component

$$\rho^{\mathsf{Sets}_*,\lhd}_X \colon X \to X \lhd_{\mathsf{Sets}_*} S^0$$

at X is given by the composition

$$\begin{aligned} X &\to X \vee X \\ &\cong |S^0| \odot X \\ &\cong S^0 \lhd_{\mathsf{Sets}_*}, \end{aligned}$$

where $\mathrm{inj}_1 \colon X \to X \vee X$ is the map sending X to the first factor of X in $X \vee X$. In other words, $\rho^{\mathsf{Sets}_*,\lhd}_X$ acts on elements as

$$\rho^{\mathsf{Sets}_*,\lhd}_X(x) \overset{\mathrm{def}}{=} x \lhd_{\mathsf{Sets}_*} 0$$

for each $x \in X$;

is a left skew monoidal category whose skew associator is a natural isomorphism.

- The sextuple $\left(\mathsf{Sets}_*, \rhd_{\mathsf{Sets}_*}, S^0, \alpha^{\mathsf{Sets}_*,\rhd}, \lambda^{\mathsf{Sets}_*,\rhd}, \rho^{\mathsf{Sets}_*,\rhd}\right)$ consisting of
  - *The Underlying Category.* The category Sets_* of pointed sets;
  - *The Skew Monoidal Product.* The right tensor product functor

    $$\rhd_{\mathsf{Sets}_*} \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*,$$

    of Item 1;
  - *The Skew Monoidal Unit.* The functor

    $$\mathbb{1}^{\mathsf{Sets}_*} \colon \mathrm{pt} \to \mathsf{Sets}_*$$

    defined by

    $$\mathbb{1}_{\mathsf{Sets}_*} \overset{\mathrm{def}}{=} S^0;$$
  - *The Skew Associators.* The natural isomorphism

$$\alpha^{\mathsf{Sets}_*,\rhd} \colon \rhd_{\mathsf{Sets}_*} \circ \left(\mathrm{id}_{\mathsf{Sets}_*} \times \rhd_{\mathsf{Sets}_*}\right) \overset{\cong}{\Longrightarrow} \rhd_{\mathsf{Sets}_*} \circ \left(\rhd_{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}\right),$$

whose component

$$\alpha^{\mathsf{Sets}_*,\rhd}_{X,Y,Z} \colon X \rhd_{\mathsf{Sets}_*} (Y \rhd_{\mathsf{Sets}_*} Z) \xrightarrow{\cong} (X \rhd_{\mathsf{Sets}_*} Y) \rhd_{\mathsf{Sets}_*} Z$$

at (X, Y, Z) is given by the composition

$$X \rhd_{\mathsf{Sets}_*} (Y \rhd_{\mathsf{Sets}_*} Z) \overset{\mathrm{def}}{=} |X| \odot (Y \rhd_{\mathsf{Sets}_*} Z)$$

$$
\begin{aligned}
&\overset{\text{def}}{=} |X| \odot (|Y| \odot Z)\\
&\cong |X| \odot \left(\bigvee_{y\in Y}(Z, z_0)\right)\\
&\cong \bigvee_{x\in X}\left(\bigvee_{y\in Y}(Z, z_0)\right)\\
&\cong \bigvee_{(x,y)\in\bigvee_{x\in X}(Y,y_0)}(Z, z_0)\\
&\cong \left|\bigvee_{x\in X}(Y, y_0)\right| \odot Z\\
&\overset{\text{def}}{=} |X \odot Y| \odot Z\\
&\overset{\text{def}}{=} |X \rhd_{\mathsf{Sets}_*} Y| \odot Z\\
&\overset{\text{def}}{=} \left(X \rhd_{\mathsf{Sets}_*} Y\right) \rhd_{\mathsf{Sets}_*} Z
\end{aligned}
$$

where the isomorphism

$$\bigvee_{x\in X}\left(\bigvee_{y\in Y}(Z, z_0)\right) \cong \bigvee_{(x,y)\in\bigvee_{x\in X}(Y,y_0)}(Z, z_0)$$

is given by $[(x,(y,z))] \mapsto [((x,y),z)]$. In other words, $\alpha^{\mathsf{Sets}_*,\rhd}_{X,Y,Z}$ acts on elements as

$$\alpha^{\mathsf{Sets}_*,\rhd}_{X,Y,Z}\left(x \rhd_{\mathsf{Sets}_*} \left(y \rhd_{\mathsf{Sets}_*} z\right)\right) \overset{\text{def}}{=} \left(x \rhd_{\mathsf{Sets}_*} y\right) \rhd_{\mathsf{Sets}_*} z$$

for each $x \rhd_{\mathsf{Sets}_*} \left(y \rhd_{\mathsf{Sets}_*} z\right) \in X \rhd_{\mathsf{Sets}_*} \left(Y \rhd_{\mathsf{Sets}_*} Z\right)$;

- *The Skew Left Unitors.* The natural transformation

$$\lambda^{\mathsf{Sets}_*,\rhd}\colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \rhd_{\mathsf{Sets}_*} \circ \left(\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}\right),$$

whose component

$$\lambda^{\mathsf{Sets}_*,\rhd}_{X}\colon X \to S^0 \rhd_{\mathsf{Sets}_*} X$$

at X is given by the composition

$$X \to X \vee X$$

$$
\begin{aligned}
&\cong |S^0| \odot X\\
&\cong S^0 \rhd_{\mathsf{Sets}_*} X,
\end{aligned}
$$

where $\mathrm{inj}_1 \colon X \to X \vee X$ is the map sending X to the first factor of X in $X \vee X$. In other words, $\lambda_X^{\mathsf{Sets}_*,\rhd}$ acts on elements as

$$\lambda_X^{\mathsf{Sets}_*,\rhd}(x) \stackrel{\text{def}}{=} 0 \rhd_{\mathsf{Sets}_*} x$$

for each $x \in X$;

- *The Skew Right Unitors.* The natural transformation

$$\rho^{\mathsf{Sets}_*,\rhd} \colon \rhd_{\mathsf{Sets}_*} \circ \left(\mathrm{id}_{\mathsf{Sets}_*} \times \mathbb{1}^{\mathsf{Sets}_*}\right) \Longrightarrow \mathrm{id}_{\mathsf{Sets}_*},$$

  whose component

$$\rho_X^{\mathsf{Sets}_*,\rhd} \colon X \rhd_{\mathsf{Sets}_*} S^0 \to X$$

  at X is given by the composition

$$
\begin{aligned}
X \rhd_{\mathsf{Sets}_*} S^0 &\cong |X| \odot S^0\\
&\cong \bigvee_{x \in X} S^0\\
&\to X
\end{aligned}
$$

  where $\bigvee_{x \in X} S^0 \to X$ is the map given by

$$
\begin{aligned}
(x, 0) &\mapsto x,\\
(x, 1) &\mapsto x.
\end{aligned}
$$

  In other words, $\rho_X^{\mathsf{Sets}_*,\lhd}$ acts on elements as

$$
\begin{aligned}
\rho_X^{\mathsf{Sets}_*,\lhd}(x \rhd_{\mathsf{Sets}_*} 0) &\stackrel{\text{def}}{=} x,\\
\rho_X^{\mathsf{Sets}_*,\lhd}(x \rhd_{\mathsf{Sets}_*} 1) &\stackrel{\text{def}}{=} x
\end{aligned}
$$

  for each $x \in X$;

is a right skew monoidal category whose skew associator is a natural isomorphism.

Proof 4.8.5 ▶ Proof of Proposition 4.8.4

Item 1: Functoriality

Omitted.

Item 2: Skew Monoidality

Omitted.

Appendices

A Other Chapters

Logic and Model Theory

1. Logic
2. Model Theory

Type Theory

3. Type Theory
4. Homotopy Type Theory

Set Theory

5. Sets
6. Constructions With Sets
7. Indexed and Fibred Sets
8. Relations
9. Posets

Category Theory

10. Categories
11. Constructions With Categories
12. Limits and Colimits
13. Ends and Coends
14. Kan Extensions
15. Fibred Categories
16. Weighted Category Theory

Categorical Hochschild Co/Homology

17. Abelian Categorical Hochschild Co/Homology
18. Categorical Hochschild Co/Homology

Monoidal Categories

19. Monoidal Categories
20. Monoidal Fibrations
21. Modules Over Monoidal Categories
22. Monoidal Limits and Colimits
23. Monoids in Monoidal Categories
24. Modules in Monoidal Categories
25. Skew Monoidal Categories
26. Promonoidal Categories
27. 2-Groups
28. Duoidal Categories
29. Semiring Categories

Categorical Algebra

30. Monads
31. Algebraic Theories
32. Coloured Operads
33. Enriched Coloured Operads

Enriched Category Theory

34. Enriched Categories
35. Enriched Ends and Kan Extensions

36. Fibred Enriched Categories
37. Weighted Enriched Category Theory

Internal Category Theory

38. Internal Categories
39. Internal Fibrations
40. Locally Internal Categories
41. Non-Cartesian Internal Categories
42. Enriched-Internal Categories

Homological Algebra

43. Abelian Categories
44. Triangulated Categories
45. Derived Categories

Categorical Logic

46. Categorical Logic
47. Elementary Topos Theory
48. Non-Cartesian Topos Theory

Sites, Sheaves, and Stacks

49. Sites
50. Modules on Sites
51. Topos Theory
52. Cohomology in a Topos
53. Stacks

Complements on Sheaves

54. Sheaves of Monoids

Bicategories

55. Bicategories
56. Biadjunctions and Pseudomonads
57. Bilimits and Bicolimits
58. Biends and Bicoends
59. Fibred Bicategories
60. Monoidal Bicategories
61. Pseudomonoids in Monoidal Bicategories

Higher Category Theory

62. Tricategories
63. Gray Monoids and Gray Categories
64. Double Categories
65. Formal Category Theory
66. Enriched Bicategories
67. Elementary 2-Topos Theory

Simplicial Stuff

68. The Simplex Category
69. Simplicial Objects
70. Cosimplicial Objects
71. Bisimplicial Objects
72. Simplicial Homotopy Theory
73. Cosimplicial Homotopy Theory

Cyclic Stuff

74. The Cycle Category
75. Cyclic Objects

Cubical Stuff

76. The Cube Category
77. Cubical Objects
78. Cubical Homotopy Theory

Globular Stuff

79. The Globe Category
80. Globular Objects

Cellular Stuff

81. The Cell Category
82. Cellular Objects

Homotopical Algebra

83. Model Categories
84. Examples of Model Categories
85. Homotopy Limits and Colimits
86. Homotopy Ends and Coends
87. Derivators

Topological and Simplicial Categories

88. Topologically Enriched Categories
89. Simplicial Categories
90. Topological Categories

Quasicategories

91. Quasicategories
92. Constructions With Quasicategories
93. Fibrations of Quasicategories
94. Limits and Colimits in Quasicategories
95. Ends and Coends in Quasicategories
96. Weighted ∞-Category Theory
97. ∞-Topos Theory

Cubical Quasicategories

98. Cubical Quasicategories

Complete Segal Spaces

99. Complete Segal Spaces

∞-Cosmoi

100. ∞-Cosmoi

Enriched and Internal ∞-Category Theory

101. Internal ∞-Categories
102. Enriched ∞-Categories

$(\infty, 2)$-Categories

103. $(\infty, 2)$-Categories
104. 2-Quasicategories

(∞, n)-Categories

105. Complicial Sets
106. Comical Sets

Double ∞-Categories

107. Double ∞-Categories

Higher Algebra

108. Differential Graded Categories
109. Stable ∞-Categories
110. ∞-Operads
111. Monoidal ∞-Categories
112. Monoids in Symmetric Monoidal ∞-Categories
113. Modules in Symmetric Monoidal ∞-Categories
114. Dendroidal Sets

Derived Algebraic Geometry

115. Derived Algebraic Geometry
116. Spectral Algebraic Geometry

Condensed Mathematics

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