Categories

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Introduction

This chapter contains basic material about categories, functors, natural transformations, adjunctions, the Yoneda Lemma, monomorphisms, and epimorphisms.

Notes to Myself

TODO:

1. Adjoints to the Yoneda embedding

Contents

1 Categories

1.1 Foundations

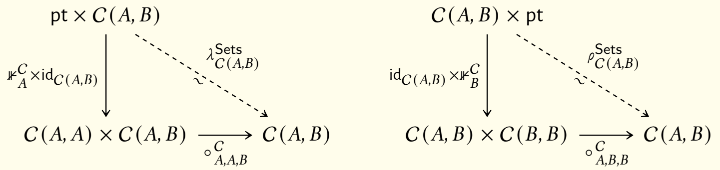
Definition 1.1.1 ▶ Categories

A **category** $(\mathcal{C}, \circ^{\mathcal{C}}, \mathbb{1}^{\mathcal{C}})$ consists of[1,2]

- *Objects.* A class $\mathrm{Obj}(\mathcal{C})$ of **objects**;
- *Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, a class $\mathrm{Hom}_{\mathcal{C}}(A, B)$, called the **class of morphisms of $\mathcal{C}$ from A to B**;
- *Identities.* For each $A \in \mathrm{Obj}(\mathcal{C})$, a map of sets
$$\mathbb{1}^{\mathcal{C}}_A \colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A, A),$$
called the **unit map of $\mathcal{C}$ at A**, determining a morphism
$$\mathrm{id}_A \colon A \longrightarrow A$$
of $\mathcal{C}$, called the **identity morphism of A**;
- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathcal{C})$, a map of sets
$$\circ^{\mathcal{C}}_{A,B,C} \colon \mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A, C),$$
called the **composition map of $\mathcal{C}$ at (A, B, C)**;

such that the following conditions are satisfied:

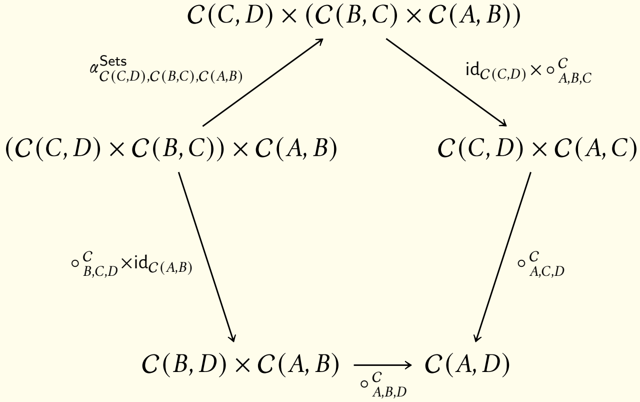
1. *Unitality of Composition.* The diagrams



commute, i.e. for each morphism $f\colon A \longrightarrow B$ of $\mathcal{C}$, we have

$$\mathrm{id}_B \circ f = f$$
$$f \circ \mathrm{id}_A = f.$$

2. *Associativity of Composition*. The diagram



commutes, i.e. for each composable triple (f, g, h) of morphisms of $\mathcal{C}$, we have

$$(f \circ g) \circ h = f \circ (g \circ h).$$

[1] *Further Notation:* We also write $\mathcal{C}(A,B)$ for $\mathrm{Hom}_{\mathcal{C}}(A,B)$.
[2] *Further Notation:* We write $\mathrm{Mor}(\mathcal{C})$ for the class of all morphisms of $\mathcal{C}$.

Definition 1.1.2 ▶ Size Conditions on Categories

Let κ be a regular cardinal. A category $\mathcal{C}$ is

1. **Locally small** if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the class $\mathrm{Hom}_{\mathcal{C}}(A,B)$ is a set;

2. **Locally essentially small** if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the class

$$\mathrm{Hom}_{\mathcal{C}}(A,B)/\{\text{isomorphisms}\}$$

is a set;

3. **Small** if $\mathcal{C}$ is locally small and $\mathrm{Obj}(\mathcal{C})$ is a set;

4. κ-**Small** if $\mathcal{C}$ is locally small, $\mathrm{Obj}(\mathcal{C})$ is a set, and

$$|\mathrm{Obj}(\mathcal{C})| < \kappa.$$

Example 1.1.3 ▶ The Punctual Category

The **punctual category**[1] is the category pt where

- *Objects.* We have

$$\mathrm{Obj}(\mathrm{pt}) \overset{\mathrm{def}}{=} \{\star\};$$

- *Morphisms.* The unique Hom-set of pt is defined by

$$\mathrm{Hom}_{\mathrm{pt}}(\star, \star) \overset{\mathrm{def}}{=} \{\mathrm{id}_\star\};$$

- *Identities.* The unit map

$$\mathbb{1}^{\mathrm{pt}}_{\star} \colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathrm{pt}}(\star, \star)$$

  of pt at $\star$ is defined by

$$\mathrm{id}^{\mathrm{pt}}_{\star} \overset{\mathrm{def}}{=} \mathrm{id}_\star;$$

- *Composition.* The composition map

$$\circ^{\mathrm{pt}}_{\star,\star,\star} \colon \mathrm{Hom}_{\mathrm{pt}}(\star, \star) \times \mathrm{Hom}_{\mathrm{pt}}(\star, \star) \longrightarrow \mathrm{Hom}_{\mathrm{pt}}(\star, \star)$$

  of pt at $(\star, \star, \star)$ is given by the bijection $\mathrm{pt} \times \mathrm{pt} \cong \mathrm{pt}$.

[1] *Further Terminology:* Also called the **singleton category**.

Example 1.1.4 ▶ Monoids as One-Object Categories

We have an isomorphism of categories[1]

$$\mathrm{Mon} \cong \mathrm{pt} \underset{\mathsf{Sets}}{\times} \mathsf{Cats}, \qquad \begin{array}{ccc} \mathrm{Mon} & \longrightarrow & \mathsf{Cats} \\ \big\downarrow & \lrcorner & \big\downarrow {\scriptstyle \mathrm{Obj}} \\ \mathrm{pt} & \underset{[\mathrm{pt}]}{\longrightarrow} & \mathsf{Sets} \end{array}$$

via the delooping functor $\mathbf{B} \colon \mathrm{Mon} \longrightarrow \mathsf{Cats}$ of **??** of **??**.

[1]This can be enhanced to an isomorphism of 2-categories

$$\mathsf{Mon}_{2\text{-disc}} \cong \mathsf{pt}_{\mathsf{bi}} \underset{\mathsf{Sets}_{2\text{-disc}}}{\times} \mathsf{Cats}_{2,*}, \qquad \begin{array}{ccc} \mathsf{Mon}_{2\text{-disc}} & \longrightarrow & \mathsf{Cats}_{2,*} \\ \downarrow & \lrcorner & \downarrow \mathsf{Obj} \\ \mathsf{pt}_{\mathsf{bi}} & \xrightarrow[{[\mathsf{pt}]}]{} & \mathsf{Sets}_{2\text{-disc}} \end{array}$$

between the discrete 2-category $\mathsf{Mon}_{2\text{-disc}}$ on Mon and the 2-category of pointed categories with one object.

Example 1.1.5 ▶ The Empty Category

The **empty category** is the category $\emptyset_{\mathsf{cat}}$ where

- *Objects.* We have
$$\mathrm{Obj}(\emptyset_{\mathsf{cat}}) \stackrel{\text{def}}{=} \emptyset;$$
- *Morphisms.* We have
$$\mathrm{Mor}(\emptyset_{\mathsf{cat}}) \stackrel{\text{def}}{=} \emptyset;$$
- *Identities and Composition.* Having no objects, $\emptyset_{\mathsf{cat}}$ has no unit nor composition maps.

Example 1.1.6 ▶ Ordinal Categories

The ***n*th ordinal category** is the category $\ltimes$ where[1]

- *Objects.* We have
$$\mathrm{Obj}(\ltimes) \stackrel{\text{def}}{=} \{[0], \ldots, [n]\};$$
- *Morphisms.* For each $[i], [j] \in \mathrm{Obj}(\ltimes)$, we have
$$\mathrm{Hom}_{\ltimes}([i],[j]) \stackrel{\text{def}}{=} \begin{cases} \{\mathrm{id}_{[i]}\} & \text{if } [i] = [j], \\ \{[i] \longrightarrow [j]\} & \text{if } [j] < [i], \\ \emptyset & \text{if } [j] > [i]; \end{cases}$$
- *Identities.* For each $[i] \in \mathrm{Obj}(\ltimes)$, the unit map
$$\nVdash^{\ltimes}_{[i]} \colon \mathsf{pt} \longrightarrow \mathrm{Hom}_{\ltimes}([i],[i])$$

of $\ltimes$ at $[i]$ is defined by

$$\mathrm{id}^{\ltimes}_{[i]} \overset{\text{def}}{=} \mathrm{id}_{[i]};$$

- *Composition.* For each $[i], [j], [k] \in \mathrm{Obj}(\ltimes)$, the composition map

$$\circ^{\ltimes}_{[i],[j],[k]} \colon \mathrm{Hom}_{\ltimes}([j],[k]) \times \mathrm{Hom}_{\ltimes}([i],[j]) \longrightarrow \mathrm{Hom}_{\ltimes}([i],[k])$$

of $\ltimes$ at $([i],[j],[k])$ is defined by

$$\begin{gathered}\mathrm{id}_{[i]} \circ \mathrm{id}_{[i]} = \mathrm{id}_{[i]},\\ ([j] \longrightarrow [k]) \circ ([i] \longrightarrow [j]) = ([i] \longrightarrow [k]).\end{gathered}$$

[1] In other words, $\ltimes$ is the category associated to the poset

$$[0] \longrightarrow [1] \longrightarrow \cdots \longrightarrow [n-1] \longrightarrow [n].$$

The category $\ltimes$ for $n \geq 2$ may also be defined in terms of $\mathit{0}$ and joins: we have isomorphisms of categories

$$\begin{aligned}\mathit{1} &\cong \mathit{0} \star \mathit{0},\\ \mathit{2} &\cong \mathit{1} \star \mathit{0}\\ &\cong (\mathit{0} \star \mathit{0}) \star \mathit{0},\\ \mathit{3} &\cong \mathit{2} \star \mathit{0}\\ &\cong (\mathit{1} \star \mathit{0}) \star \mathit{0}\\ &\cong ((\mathit{0} \star \mathit{0}) \star \mathit{0}) \star \mathit{0},\\ \mathit{4} &\cong \mathit{3} \star \mathit{0}\\ &\cong (\mathit{2} \star \mathit{0}) \star \mathit{0}\\ &\cong ((\mathit{1} \star \mathit{0}) \star \mathit{0}) \star \mathit{0}\\ &\cong (((\mathit{0} \star \mathit{0}) \star \mathit{0}) \star \mathit{0}) \star \mathit{0},\end{aligned}$$

and so on.

1.2 Subcategories

Let $\mathcal{C}$ be a category.

Definition 1.2.1 ▶ Subcategories

A **subcategory** of $\mathcal{C}$ is a category $\mathcal{A}$ satisfying the following conditions:

1. *Objects.* We have $\mathrm{Obj}(\mathcal{A}) \subset \mathrm{Obj}(\mathcal{C})$.

2. *Morphisms*. For each $A, B \in \mathrm{Obj}(\mathcal{A})$, we have

$$\mathrm{Hom}_{\mathcal{A}}(A, B) \subset \mathrm{Hom}_{\mathcal{C}}(A, B).$$

3. *Identities*. For each $A \in \mathrm{Obj}(\mathcal{A})$, we have

$$\mathbb{1}^{\mathcal{A}}_{A} = \mathbb{1}^{\mathcal{C}}_{A}.$$

4. *Composition*. For each $A, B, C \in \mathrm{Obj}(\mathcal{A})$, we have

$$\circ^{\mathcal{A}}_{A,B,C} = \circ^{\mathcal{C}}_{A,B,C}.$$

Definition 1.2.2 ▶ Full Subcategories

A subcategory $\mathcal{A}$ of $\mathcal{C}$ is **full** if the canonical inclusion functor $\mathcal{A} \longrightarrow \mathcal{C}$ is full.

Definition 1.2.3 ▶ Strictly Full Subcategories

A subcategory $\mathcal{A}$ of a category $\mathcal{C}$ is **strictly full** if it satisfies the following conditions:

1. *Fullness*. The subcategory $\mathcal{A}$ is full.

2. *Closedness Under Isomorphisms*. The class $\mathrm{Obj}(\mathcal{A})$ is closed under isomorphisms[1].

[1]That is, given $A \in \mathrm{Obj}(\mathcal{A})$ and $C \in \mathrm{Obj}(\mathcal{C})$ with $C \cong A$, we have $C \in \mathrm{Obj}(\mathcal{A})$.

Definition 1.2.4 ▶ Wide Subcategories

A subcategory $\mathcal{A}$ of $\mathcal{C}$ is **wide**[1] if $\mathrm{Obj}(\mathcal{A}) = \mathrm{Obj}(\mathcal{C})$.

[1]*Further Terminology:* Or **lluf**.

1.3 Skeletons of Categories

Definition 1.3.1 ▶ Skeletons of Categories

A[1] **skeleton** of a category $\mathcal{C}$ is a full subcategory $\mathrm{Sk}(\mathcal{C})$ with one object from each isomorphism class of objects of $\mathcal{C}$.

[1]Due to Item 2 of Proposition 1.3.3, we often refer to any such full subcategory $\mathsf{Sk}(\mathcal{C})$ of $\mathcal{C}$ as *the* skeleton of $\mathcal{C}$.

Definition 1.3.2 ▶ Skeletal Categories

A category $\mathcal{C}$ is **skeletal** if $\mathcal{C} \cong \mathsf{Sk}(\mathcal{C})$.[1]

[1]That is, $\mathcal{C}$ is **skeletal** if isomorphic objects of $\mathcal{C}$ are equal.

Proposition 1.3.3 ▶ Properties of Skeletons of Categories

Let $\mathcal{C}$ be a category.

1. *Pseudofunctoriality.* The assignment $\mathcal{C} \mapsto \mathsf{Sk}(\mathcal{C})$ defines a pseudofunctor

$$\mathsf{Sk}\colon \mathsf{Cats}_2 \longrightarrow \mathsf{Cats}_2.$$

2. *Uniqueness Up to Equivalence.* Any two skeletons of $\mathcal{C}$ are equivalent.
3. *Inclusions of Skeletons Are Equivalences.* The $\mathsf{Sk}(\mathcal{C}) \hookrightarrow \mathcal{C}$ of a skeleton of $\mathcal{C}$ into $\mathcal{C}$ is an equivalence of categories.

Proof 1.3.4 ▶ Proof of Proposition 1.3.3

Item 1: Pseudofunctoriality

See [nLab23d, Skeletons as an Endo-Pseudofunctor on $\mathfrak{Cat}$].

Item 2: Uniqueness Up to Equivalence

Clear.

Item 3: Inclusions of Skeletons Are Equivalences

Clear. □

1.4 Precomposition and Postcomposition

Let $\mathcal{C}$ be a category, let $A, B, C \in \mathrm{Obj}(\mathcal{C})$, and let $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$ be morphisms of $\mathcal{C}$.

Definition 1.4.1 ▶ Precomposition

The **precomposition function associated to** f is the function

$$f^*\colon \operatorname{Hom}_{\mathcal{C}}(B, C) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(A, C)$$

defined by

$$f^*(\phi) \stackrel{\text{def}}{=} \phi \circ f$$

for each $\phi \in \operatorname{Hom}_{\mathcal{C}}(B, C)$.

Definition 1.4.2 ▶ Postcomposition

The **postcomposition function associated to** g is the function

$$g_*\colon \operatorname{Hom}_{\mathcal{C}}(A, B) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(A, C)$$

defined by

$$g_*(\phi) \stackrel{\text{def}}{=} g \circ \phi$$

for each $\phi \in \operatorname{Hom}_{\mathcal{C}}(A, B)$.

Proposition 1.4.3 ▶ Properties of Pre/Postcomposition

Let $A, B, C, D \in \operatorname{Obj}(\mathcal{C})$ and let $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$ be morphisms of $\mathcal{C}$.

1. *Interaction Between Precomposition and Postcomposition.* We have

$$g_* \circ f^* = f^* \circ g_*, \qquad \begin{array}{ccc} \operatorname{Hom}_{\mathcal{C}}(B, C) & \xrightarrow{g_*} & \operatorname{Hom}_{\mathcal{C}}(B, D) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \operatorname{Hom}_{\mathcal{C}}(A, C) & \xrightarrow[g_*]{} & \operatorname{Hom}_{\mathcal{C}}(A, D). \end{array}$$

2. *Interaction With Composition I.* We have

$$(g \circ f)^* = f^* \circ g^*, \qquad \begin{array}{ccc} \operatorname{Hom}_{\mathcal{C}}(X, A) & \xrightarrow{f_*} & \operatorname{Hom}_{\mathcal{C}}(X, B) \\ & {\scriptstyle (g \circ f)_*}\searrow & \downarrow{\scriptstyle g_*} \\ & & \operatorname{Hom}_{\mathcal{C}}(X, C), \end{array}$$

$$(g \circ f)_* = g_* \circ f_*, \qquad \begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(C, X) & \xrightarrow{g^*} & \mathrm{Hom}_{\mathcal{C}}(B, X) \\ & \searrow^{(g\circ f)^*} & \downarrow{\scriptstyle f^*} \\ & & \mathrm{Hom}_{\mathcal{C}}(A, X). \end{array}$$

3. *Interaction With Composition II.* We have

$$\begin{array}{ccc} \mathrm{pt} & \xrightarrow{[f]} & \mathrm{Hom}_{\mathcal{C}}(A, B) \\ & \searrow^{[g\circ f]} & \downarrow{\scriptstyle g_*} \\ & & \mathrm{Hom}_{\mathcal{C}}(A, C) \end{array} \qquad \begin{aligned} [g \circ f] &= g_* \circ [f], \\ [g \circ f] &= f^* \circ [g], \end{aligned} \qquad \begin{array}{ccc} \mathrm{pt} & \xrightarrow{[g]} & \mathrm{Hom}_{\mathcal{C}}(B, C) \\ & \searrow^{[g\circ f]} & \downarrow{\scriptstyle f^*} \\ & & \mathrm{Hom}_{\mathcal{C}}(A, C). \end{array}$$

4. *Interaction With Composition III.* We have

$$f^* \circ \circ^{\mathcal{C}}_{A,B,C} = \circ^{\mathcal{C}}_{X,B,C} \circ (f^* \times \mathrm{id}), \qquad \begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) & \xrightarrow{\circ^{\mathcal{C}}_{A,B,C}} & \mathrm{Hom}_{\mathcal{C}}(A, C) \\ \downarrow{\scriptstyle f^* \times \mathrm{id}} & & \downarrow{\scriptstyle f^*} \\ \mathrm{Hom}_{\mathcal{C}}(X, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) & \xrightarrow[\circ^{\mathcal{C}}_{X,B,C}]{} & \mathrm{Hom}_{\mathcal{C}}(X, C), \end{array}$$

$$g_* \circ \circ^{\mathcal{C}}_{A,B,C} = \circ^{\mathcal{C}}_{A,B,D} \circ (\mathrm{id} \times g_*), \qquad \begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) & \xrightarrow{\circ^{\mathcal{C}}_{A,B,C}} & \mathrm{Hom}_{\mathcal{C}}(A, C) \\ \downarrow{\scriptstyle \mathrm{id} \times g_*} & & \downarrow{\scriptstyle g^*} \\ \mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, D) & \xrightarrow[\circ^{\mathcal{C}}_{A,B,D}]{} & \mathrm{Hom}_{\mathcal{C}}(A, D). \end{array}$$

5. *Interaction With Identities.* We have

$$\begin{aligned} (\mathrm{id}_A)^* &= \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}, \\ (\mathrm{id}_B)_* &= \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}. \end{aligned}$$

Proof 1.4.4 ▶ Proof of Proposition 1.4.3

Item 1: Interaction Between Precomposition and Postcomposition

Omitted.

Item 2: Interaction With Composition I

Omitted.

Item 3: Interaction With Composition II

Omitted.

Item 4: Interaction With Composition III

Omitted.

Item 5: Interaction With Identities

Omitted.

1.5 The Fundamental Quadruple Adjunction

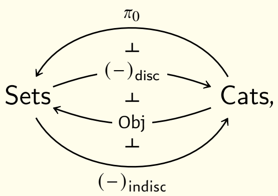
1.5.1 Statement

Let $\mathcal{C}$ be a category.

Proposition 1.5.1 ▶ A Quadruple Adjunction Between Sets and Cats

We have a quadruple adjunction

$$(\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathsf{Obj} \dashv (-)_{\mathsf{indisc}})\colon$$



witnessed by bijections of sets

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{Sets}}(\pi_0(\mathcal{C}), X) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, X_{\mathsf{disc}}),\\
\mathrm{Hom}_{\mathsf{Cats}}(X_{\mathsf{disc}}, \mathcal{C}) &\cong \mathrm{Hom}_{\mathsf{Sets}}(X, \mathsf{Obj}(\mathcal{C})),\\
\mathrm{Hom}_{\mathsf{Sets}}(\mathsf{Obj}(\mathcal{C}), X) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, X_{\mathsf{indisc}}),
\end{aligned}$$

natural in $\mathcal{C} \in \mathsf{Obj}(\mathsf{Cats})$ and $X \in \mathsf{Obj}(\mathsf{Sets})$, where

- π_0, the **connected components functor**, is the functor sending a category $\mathcal{C}$ to the set $\pi_0(\mathcal{C})$ of connected components of $\mathcal{C}$ of Definition 1.5.4;
- $(-)_{\mathsf{disc}}$, the **discrete category functor** is the functor sending a set X to the discrete category X_{disc} associated to X of Definition 1.5.8;

- Obj is the functor sending a category to its set of objects;
- $(-)_{\text{indisc}}$, the **indiscrete category functor** is the functor sending a set X to the indiscrete category X_{indisc} associated to X of Definition 1.5.11.

PROOF 1.5.2 ▶ PROOF OF PROPOSITION 1.5.1

Omitted.

1.5.2 Connected Components of Categories

Let $\mathcal{C}$ be a category.

DEFINITION 1.5.3 ▶ CONNECTED COMPONENTS OF CATEGORIES

A **connected component** of $\mathcal{C}$ is a full subcategory $\mathcal{I}$ of $\mathcal{C}$ satisfying the following conditions:[1]

1. *Non-Emptiness.* We have $\text{Obj}(\mathcal{I}) \neq \emptyset$.
2. *Connectedness.* There exists a zigzag of arrows between any two objects of $\mathcal{I}$.

[1] In other words, a **connected component** of $\mathcal{C}$ is an element of the set $\text{Obj}(\mathcal{C})/\sim$ with $\sim$ the equivalence relation generated by the relation $\sim'$ obtained by declaring $A \sim' B$ iff there exists a morphism of $\mathcal{C}$ from A to B.

1.5.3 Sets of Connected Components of Categories

Let $\mathcal{C}$ be a category.

DEFINITION 1.5.4 ▶ SETS OF CONNECTED COMPONENTS OF CATEGORIES

The **set of connected components of** $\mathcal{C}$ is the set $\pi_0(\mathcal{C})$ whose elements are the connected components of $\mathcal{C}$.

PROPOSITION 1.5.5 ▶ PROPERTIES OF SETS OF CONNECTED COMPONENTS

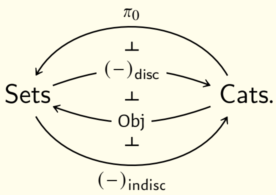
Let $\mathcal{C}$ be a category.

1. *Functoriality.* The assignment $\mathcal{C} \mapsto \pi_0(\mathcal{C})$ defines a functor

$$\pi_0 \colon \mathsf{Cats} \longrightarrow \mathsf{Sets}.$$

2. *Adjointness*[1]. We have a quadruple adjunction

$$(\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathsf{Obj} \dashv (-)_{\mathsf{indisc}})\colon$$



3. *Interaction With Groupoids.* If $\mathcal{C}$ is a groupoid, then we have an isomorphism of categories

$$\pi_0(\mathcal{C}) \cong \mathsf{K}(\mathcal{C}).$$

4. *Preservation of Colimits.* The functor π_0 of Item 1 preserves colimits. In particular, we have bijections of sets

$$\begin{aligned}\pi_0\left(\mathcal{C} \coprod \mathcal{D}\right) &\cong \pi_0(\mathcal{C}) \coprod \pi_0(\mathcal{D}),\\ \pi_0\left(\mathcal{C} \textstyle\coprod_{\mathcal{E}} \mathcal{D}\right) &\cong \pi_0(\mathcal{C}) \textstyle\coprod_{\pi_0(\mathcal{E})} \pi_0(\mathcal{D}),\\ \pi_0\left(\mathsf{CoEq}\left(\mathcal{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{D}\right)\right) &\cong \mathsf{CoEq}\left(\pi_0(\mathcal{C}) \underset{\pi_0(G)}{\overset{\pi_0(F)}{\rightrightarrows}} \pi_0(\mathcal{D})\right),\end{aligned}$$

natural in $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathsf{Obj}(\mathsf{Cats})$.

5. *Symmetric Strong Monoidality With Respect to Coproducts.* The connected components functor of Item 1 has a symmetric strong monoidal structure

$$\left(\pi_0, \pi_0^{\coprod}, \pi_{0|\mathbb{1}}^{\coprod}\right)\colon (\mathsf{Cats}, \textstyle\coprod, \emptyset_{\mathsf{cat}}) \longrightarrow (\mathsf{Sets}, \textstyle\coprod, \emptyset),$$

being equipped with isomorphisms

$$\begin{aligned}\pi_{0|\mathcal{C},\mathcal{D}}^{\coprod}\colon \pi_0(\mathcal{C}) \textstyle\coprod \pi_0(\mathcal{D}) &\xrightarrow{\cong} \pi_0\left(\mathcal{C} \textstyle\coprod \mathcal{D}\right),\\ \pi_{0|\mathbb{1}}^{\coprod}\colon \emptyset &\xrightarrow{\cong} \pi_0(\emptyset_{\mathsf{cat}}),\end{aligned}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathsf{Obj}(\mathsf{Cats})$.

6. *Symmetric Strong Monoidality With Respect to Products.* The connected components functor of Item 1 has a symmetric strong monoidal structure

$$\left(\pi_0, \pi_0^{\otimes}, \pi_{0|\mathbb{1}}^{\otimes}\right)\colon (\mathsf{Cats}, \times, \mathsf{pt}) \longrightarrow (\mathsf{Sets}, \times, \mathsf{pt}),$$

being equipped with isomorphisms

$$\pi^{\otimes}_{0|\mathcal{C},\mathcal{D}} \colon \pi_0(\mathcal{C}) \times \pi_0(\mathcal{D}) \xrightarrow{\cong} \pi_0(\mathcal{C} \times \mathcal{D}),$$
$$\pi^{\otimes}_{0|\mathbb{1}} \colon \mathrm{pt} \xrightarrow{\cong} \pi_0(\mathrm{pt}),$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

[1] This is a repetition of Proposition 1.5.1.

Proof 1.5.6 ▶ Proof of Proposition 1.5.5

Item 1: Functoriality

Omitted.

Item 2: Adjointness

Omitted.

Item 3: Interaction With Groupoids

Clear.

Item 4: Preservation of Colimits

This follows from Item 2 and Item 4 of Proposition 6.1.3.

Item 5: Symmetric Strong Monoidality With Respect to Coproducts

Omitted.

Item 6: Symmetric Strong Monoidality With Respect to Products

Omitted. □

1.5.4 Connected Categories

Definition 1.5.7 ▶ Connected Categories

A category $\mathcal{C}$ is **connected** if $\pi_0(\mathcal{C}) \cong \mathrm{pt}$.[1,2]

[1] *Further Terminology:* Moreover, a category is **disconnected** if it is not connected.
[2] *Example:* A groupoid is connected iff any two of its objects are isomorphic.

1.5.5 Discrete Categories

Let X be a set.

Definition 1.5.8 ▶ The Discrete Category on a Set

The **discrete category on a set** X is the category X_{disc} where

- *Objects.* We have
$$\mathrm{Obj}(X_{\mathsf{disc}}) \overset{\text{def}}{=} X;$$
- *Morphisms.* For each $A, B \in \mathrm{Obj}(X_{\mathsf{disc}})$, we have
$$\mathrm{Hom}_{X_{\mathsf{disc}}}(A, B) \overset{\text{def}}{=} \begin{cases} \mathrm{id}_A & \text{if } A = B, \\ \emptyset & \text{if } A \neq B; \end{cases}$$
- *Identities.* For each $A \in \mathrm{Obj}(X_{\mathsf{disc}})$, the unit map
$$\mathbb{1}^{X_{\mathsf{disc}}}_A \colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{X_{\mathsf{disc}}}(A, A)$$
of X_{disc} at A is defined by
$$\mathrm{id}^{X_{\mathsf{disc}}}_A \overset{\text{def}}{=} \mathrm{id}_A;$$
- *Composition.* For each $A, B, C \in \mathrm{Obj}(X_{\mathsf{disc}})$, the composition map
$$\circ^{X_{\mathsf{disc}}}_{A,B,C} \colon \mathrm{Hom}_{X_{\mathsf{disc}}}(B, C) \times \mathrm{Hom}_{X_{\mathsf{disc}}}(A, B) \longrightarrow \mathrm{Hom}_{X_{\mathsf{disc}}}(A, C)$$
of X_{disc} at (A, B, C) is defined by
$$\mathrm{id}_A \circ \mathrm{id}_A \overset{\text{def}}{=} \mathrm{id}_A.$$

Proposition 1.5.9 ▶ Properties of Discrete Categories on Sets

Let X be a set.

1. *Functoriality.* The assignment $X \mapsto X_{\mathsf{disc}}$ defines a functor
$$(-)_{\mathsf{disc}} \colon \mathsf{Sets} \longrightarrow \mathsf{Cats}.$$
2. *Symmetric Strong Monoidality With Respect to Coproducts.* The functor of Item 1 has a symmetric strong monoidal structure
$$\left((-)_{\mathsf{disc}}, (-)^{\coprod}_{\mathsf{disc}}, (-)^{\coprod}_{\mathsf{disc}|\mathbb{1}}\right) \colon (\mathsf{Sets}, \textstyle\coprod, \emptyset) \longrightarrow (\mathsf{Cats}, \textstyle\coprod, \emptyset_{\mathsf{cat}}),$$

being equipped with isomorphisms

$$(-)^{\coprod}_{\mathsf{disc}|X,Y}\colon X_{\mathsf{disc}} \coprod Y_{\mathsf{disc}} \xrightarrow{\cong} (X \coprod Y)_{\mathsf{disc}},$$
$$(-)^{\coprod}_{\mathsf{disc}|\mathbb{1}}\colon \emptyset_{\mathsf{cat}} \xrightarrow{\cong} \emptyset_{\mathsf{disc}},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Symmetric Strong Monoidality With Respect to Products.* The functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)_{\mathsf{disc}}, (-)^{\otimes}_{\mathsf{disc}}, (-)^{\otimes}_{\mathsf{disc}|\mathbb{1}}\right)\colon (\mathsf{Sets}, \times, \mathrm{pt}) \longrightarrow (\mathsf{Cats}, \times, \mathrm{pt}),$$

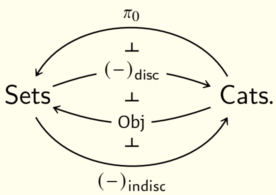
being equipped with isomorphisms

$$(-)^{\otimes}_{\mathsf{disc}|X,Y}\colon X_{\mathsf{disc}} \times Y_{\mathsf{disc}} \xrightarrow{\cong} (X \times Y)_{\mathsf{disc}},$$
$$(-)^{\otimes}_{\mathsf{disc}|\mathbb{1}}\colon \mathrm{pt} \xrightarrow{\cong} \mathrm{pt}_{\mathsf{disc}},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Adjointness*[1]. We have a quadruple adjunction

$$(\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathrm{Obj} \dashv (-)_{\mathsf{indisc}})\colon$$



[1]This is a repetition of Proposition 1.5.1.

Proof 1.5.10 ▶ Proof of Proposition 1.5.9

Item 1: Functoriality

Omitted.

Item 2: Symmetric Strong Monoidality With Respect to Coproducts

Omitted.

Item 3: Symmetric Strong Monoidality With Respect to Products

Omitted.

Item 4: Adjointness

This was proved in its repetition, Proposition 1.5.1.

1.5.6 Indiscrete Categories

Definition 1.5.11 ▶ The Indiscrete category on a Set

The **indiscrete category on a set** X[1] is the category X_{indisc} where

- *Objects.* We have
$$\mathrm{Obj}(X_{\mathsf{indisc}}) \stackrel{\mathrm{def}}{=} X;$$
- *Morphisms.* For each $A, B \in \mathrm{Obj}(X_{\mathsf{indisc}})$, we have
$$\mathrm{Hom}_{X_{\mathsf{disc}}}(A, B) \stackrel{\mathrm{def}}{=} \{[A] \longrightarrow [B]\};$$
- *Identities.* For each $A \in \mathrm{Obj}(X_{\mathsf{indisc}})$, the unit map
$$\mathbb{1}^{X_{\mathsf{indisc}}}_{A} : \mathrm{pt} \longrightarrow \mathrm{Hom}_{X_{\mathsf{indisc}}}(A, A)$$
of X_{indisc} at A is defined by
$$\mathrm{id}^{X_{\mathsf{indisc}}}_{A} \stackrel{\mathrm{def}}{=} \{[A] \longrightarrow [A]\};$$
- *Composition.* For each $A, B, C \in \mathrm{Obj}(X_{\mathsf{indisc}})$, the composition map
$$\circ^{X_{\mathsf{indisc}}}_{A,B,C} : \mathrm{Hom}_{X_{\mathsf{indisc}}}(B, C) \times \mathrm{Hom}_{X_{\mathsf{indisc}}}(A, B) \longrightarrow \mathrm{Hom}_{X_{\mathsf{indisc}}}(A, C)$$
of X_{disc} at (A, B, C) is defined by
$$([B] \longrightarrow [C]) \circ ([A] \longrightarrow [B]) \stackrel{\mathrm{def}}{=} ([A] \longrightarrow [C]).$$

[1] *Further Terminology:* Also called the **chaotic category on** X.

Proposition 1.5.12 ▶ Properties of Indiscrete Categories on Sets

Let X be a set.

1. *Functoriality.* The assignment $X \mapsto X_{\mathsf{indisc}}$ defines a functor
$$(-)_{\mathsf{indisc}} : \mathsf{Sets} \longrightarrow \mathsf{Cats}.$$

2. *Symmetric Strong Monoidality With Respect to Products.* The functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)_{\mathsf{indisc}}, (-)^{\otimes}_{\mathsf{indisc}}, (-)^{\otimes}_{\mathsf{indisc}|\mathbb{1}}\right) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \longrightarrow (\mathsf{Cats}, \times, \mathrm{pt}),$$

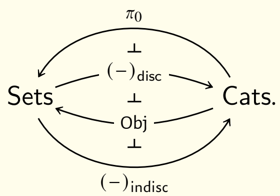
being equipped with isomorphisms

$$(-)^{\otimes}_{\mathsf{indisc}|X,Y} \colon X_{\mathsf{indisc}} \times Y_{\mathsf{indisc}} \xrightarrow{\cong} (X \times Y)_{\mathsf{indisc}},$$
$$(-)^{\otimes}_{\mathsf{indisc}|\mathbb{1}} \colon \mathrm{pt} \xrightarrow{\cong} \mathrm{pt}_{\mathsf{indisc}},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Adjointness*[1]. We have a quadruple adjunction

$(\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathrm{Obj} \dashv (-)_{\mathsf{indisc}})$:



[1]This is a repetition of Proposition 1.5.1.

Proof 1.5.13 ▶ Proof of Proposition 1.5.12

Item 1: Functoriality

Omitted.

Item 2: Symmetric Strong Monoidality With Respect to Products

Omitted.

Item 3: Adjointness

This was proved in its repetition, Proposition 1.5.1.

1.6 Groupoids

1.6.1 Foundations

Let $\mathcal{C}$ be a category.

DEFINITION 1.6.1 ▶ ISOMORPHISMS

A morphism $f\colon A \longrightarrow B$ of $\mathcal{C}$ is an **isomorphism** if there exists a morphism $f^{-1}\colon B \longrightarrow A$ of $\mathcal{C}$ such that

$$\begin{aligned} f \circ f^{-1} &= \mathrm{id}_B, \\ f^{-1} \circ f &= \mathrm{id}_A. \end{aligned}$$

DEFINITION 1.6.2 ▶ GROUPOIDS

A **groupoid** is a category in which every morphism is an isomorphism.

1.6.2 The Groupoid Completion of a Category

Let $\mathcal{C}$ be a category.

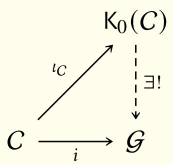
DEFINITION 1.6.3 ▶ THE GROUPOID COMPLETION OF A CATEGORY

The **groupoid completion of** $\mathcal{C}$[1] is the pair $(\mathsf{K}_0(\mathcal{C}), \iota_{\mathcal{C}})$ consisting of

- A groupoid $\mathsf{K}_0(\mathcal{C})$;
- A functor $\iota_{\mathcal{C}}\colon \mathcal{C} \longrightarrow \mathsf{K}_0(\mathcal{C})$;

satisfying the following universal property:

(UP) Given another such pair $(\mathcal{G}, i)$, there exists a unique functor $\mathsf{K}_0(\mathcal{C}) \xrightarrow{\exists!} \mathcal{G}$ making the diagram



commute.

[1] *Further Terminology:* Also called the **Grothendieck groupoid of** $\mathcal{C}$ or the **Grothendieck groupoid completion of** $\mathcal{C}$.

Proposition 1.6.4 ▶ Properties of Groupoid Completion

Let $\mathcal{C}$ be a category.

1. *Functoriality.* The assignment $\mathcal{C} \mapsto \mathrm{K}_0(\mathcal{C})$ defines a functor

$$\mathrm{K}_0 \colon \mathsf{Cats} \longrightarrow \mathsf{Grpd}.$$

2. *Adjointness.* We have an adjunction

$$(\mathrm{K}_0 \dashv \iota)\colon \quad \mathsf{Cats} \underset{\iota}{\overset{\mathrm{K}_0}{\underset{\bot}{\rightleftarrows}}} \mathsf{Grpd},$$

forming, together with the core functor Core of Item 1 of Proposition 1.6.9, a triple adjunction

$$(\mathrm{K}_0 \dashv \iota \dashv \mathrm{Core})\colon \quad \mathsf{Cats} \begin{array}{c} \xrightarrow{\mathrm{K}_0} \\ \bot \\ \xleftarrow{\iota} \\ \bot \\ \xrightarrow[\mathrm{Core}]{} \end{array} \mathsf{Grpd}.$$

3. *Interaction With Classifying Spaces.* We have an isomorphism of groupoids

$$\mathrm{K}_0(\mathcal{C}) \cong \Pi_{\leq 1}(\mathrm{B}\mathcal{C}),$$

natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$; i.e. the diagram

$$\begin{array}{ccc} \mathsf{Cats} & \xrightarrow{\mathrm{K}_0} & \mathsf{Grp} \\ {\scriptstyle \mathrm{N}_\bullet}\downarrow & \wr\Updownarrow & \uparrow{\scriptstyle \Pi_{\leq 1}} \\ \mathsf{sSets} & \xrightarrow[|-|]{} & \mathsf{Top} \end{array}$$

commutes up to natural isomorphism.

4. *Symmetric Strong Monoidality With Respect to Coproducts.* The groupoid completion functor of Item 1 has a symmetric strong monoidal structure

$$\left(\mathrm{K}_0, \mathrm{K}_0^{\coprod}, \mathrm{K}_{0|\mathbb{1}}^{\coprod}\right)\colon (\mathsf{Cats}, \coprod, \emptyset_{\mathsf{cat}}) \longrightarrow (\mathsf{Grpd}, \coprod, \emptyset_{\mathsf{cat}})$$

being equipped with isomorphisms

$$K^{\coprod}_{0|\mathcal{C},\mathcal{D}} \colon K_0(\mathcal{C}) \coprod K_0(\mathcal{D}) \xrightarrow{\cong} K_0(\mathcal{C} \coprod \mathcal{D}),$$
$$K^{\coprod}_{0|\mathbb{1}} \colon \emptyset_{\mathsf{cat}} \xrightarrow{\cong} K_0(\emptyset_{\mathsf{cat}}),$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

5. *Symmetric Strong Monoidality With Respect to Products.* The groupoid completion functor of Item 1 has a symmetric strong monoidal structure

$$\left(K_0, K_0^{\times}, K_{0|\mathbb{1}}^{\times}\right) \colon (\mathsf{Cats}, \times, \mathrm{pt}) \longrightarrow (\mathsf{Grpd}, \times, \mathrm{pt})$$

being equipped with isomorphisms

$$K^{\times}_{0|\mathcal{C},\mathcal{D}} \colon K_0(\mathcal{C}) \times K_0(\mathcal{D}) \xrightarrow{\cong} K_0(\mathcal{C} \times \mathcal{D}),$$
$$K^{\times}_{0|\mathbb{1}} \colon \mathrm{pt} \xrightarrow{\cong} K_0(\mathrm{pt}),$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

PROOF 1.6.5 ▶ PROOF OF PROPOSITION 1.6.4

Item 1: Functoriality

Omitted.

Item 2: Adjointness

Omitted.

Item 3: Interaction With Classifying Spaces

See Corollary 18.33 of `https://web.ma.utexas.edu/users/dafr/M392C-2012/Notes/lecture18.pdf`.

Item 4: Symmetric Strong Monoidality With Respect to Coproducts

Omitted.

Item 5: Symmetric Strong Monoidality With Respect to Products

Omitted. □

1.6.3 The Core of a Category

Let $\mathcal{C}$ be a category.

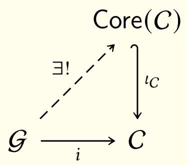
Definition 1.6.6 ▶ The Core of a Category

The **core** of $\mathcal{C}$ is the pair $(\mathrm{Core}(\mathcal{C}), \iota_{\mathcal{C}})$[1] consisting of

1. A groupoid $\mathrm{Core}(\mathcal{C})$;
2. A functor $\iota_{\mathcal{C}} \colon \mathrm{Core}(\mathcal{C}) \hookrightarrow \mathcal{C}$;

satisfying the following universal property:

(**UP**) Given another such pair $(\mathcal{G}, i)$, there exists a unique functor $\mathcal{G} \xrightarrow{\exists!} \mathrm{Core}(\mathcal{C})$ making the diagram



commute.

[1] *Further Notation:* Also written $\mathcal{C}^{\simeq}$.

Construction 1.6.7 ▶ Construction of the Core of a Category

The core of $\mathcal{C}$ is the unique subcategory of $\mathcal{C}$ where[1]

1. *Objects.* We have

$$\mathrm{Obj}(\mathrm{Core}(\mathcal{C})) \overset{\mathrm{def}}{=} \mathrm{Obj}(\mathcal{C});$$

2. *Morphisms.* The morphisms of $\mathrm{Core}(\mathcal{C})$ are the isomorphisms of $\mathcal{C}$.

[1] In other words, $\mathrm{Core}(\mathcal{C})$ is the maximal subgroupoid of $\mathcal{C}$.

Proof 1.6.8 ▶ Proof of Construction 1.6.7

This follows from the fact that functors preserve isomorphisms. □

Proposition 1.6.9 ▶ Properties of the Core of a Category

Let $\mathcal{C}$ be a category.

1. *Functoriality.* The assignment $\mathcal{C} \mapsto \mathrm{Core}(\mathcal{C})$ defines a functor

$$\mathrm{Core}\colon \mathsf{Cats} \longrightarrow \mathsf{Grpd}.$$

2. *Adjointness.* We have an adjunction

$$(\iota \dashv \mathrm{Core})\colon \quad \mathsf{Grpd} \underset{\mathrm{Core}}{\overset{\iota}{\underset{\bot}{\rightleftarrows}}} \mathsf{Cats},$$

forming, together with the groupoid completion functor K_0 of Item 1 of Proposition 1.6.4, a triple adjunction

$$(\mathrm{K}_0 \dashv \iota \dashv \mathrm{Core})\colon \quad \mathsf{Cats} \overset{\mathrm{K}_0}{\underset{\mathrm{Core}}{\rightleftarrows}} \mathsf{Grpd}, \quad \iota\colon \mathsf{Grpd} \to \mathsf{Cats}, \quad \mathrm{K}_0 \dashv \iota \dashv \mathrm{Core}.$$

3. *Symmetric Strong Monoidality With Respect to Products.* The core functor of Item 1 has a symmetric strong monoidal structure

$$\left(\mathrm{Core}, \mathrm{Core}^{\times}, \mathrm{Core}^{\times}_{\mathbb{1}}\right) \colon (\mathsf{Cats}, \times, \mathrm{pt}) \longrightarrow (\mathsf{Grpd}, \times, \mathrm{pt})$$

being equipped with isomorphisms

$$\mathrm{Core}^{\times}_{\mathcal{C},\mathcal{D}} \colon \mathrm{Core}(\mathcal{C}) \times \mathrm{Core}(\mathcal{D}) \xrightarrow{\cong} \mathrm{Core}(\mathcal{C} \times \mathcal{D}),$$

$$\mathrm{Core}^{\times}_{\mathbb{1}} \colon \mathrm{pt} \xrightarrow{\cong} \mathrm{Core}(\mathrm{pt}),$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

Proof 1.6.10 ▶ Proof of Proposition 1.6.9

Item 1: Functoriality

Clear.

Item 2: Adjointness

The adjunction $(\mathrm{K}_0 \dashv \iota)$ follows from the universal property of the Gabriel–Zisman localisation of a category with respect to a class of morphisms (**??**), while the adjunction $(\iota \dashv \mathrm{Core})$ is a reformulation of the universal property of the core of a category (Definition 1.6.6).[1]

Item 3: Symmetric Strong Monoidality With Respect to Products

Omitted.

[1] *Reference:* [Rie17, Example 4.1.15]

2 Functors and Natural Transformations

2.1 Functors

2.1.1 Foundations

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 2.1.1 ▶ Functors

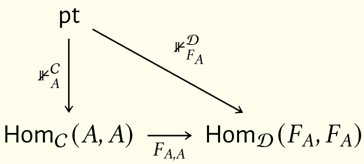
A **functor** $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ **from** $\mathcal{C}$ **to** $\mathcal{D}$[1] consists of[2]

1. *Action on Objects.* A map of sets
$$F\colon \ \mathrm{Obj}(\mathcal{C}) \longrightarrow \mathrm{Obj}(\mathcal{D}),$$
called the **action on objects of** F;
2. *Action on* Hom*-sets.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, a map
$$F_{A,B}\colon \ \mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F_A, F_B),$$
called the **action on** Hom**-sets of** F **at** (A, B);

satisfying the following conditions:

1. *Preservation of Composition.* For each $A, B, C \in \mathrm{Obj}(\mathcal{C})$, the diagram
$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) & \xrightarrow{\circ^{\mathcal{C}}_{A,B,C}} & \mathrm{Hom}_{\mathcal{C}}(A, C) \\ {\scriptstyle F_{B,C}\times F_{A,B}}\downarrow & & \downarrow{\scriptstyle F_{A,C}} \\ \mathrm{Hom}_{\mathcal{D}}(F_B, F_C) \times \mathrm{Hom}_{\mathcal{D}}(F_A, F_B) & \xrightarrow[\circ^{\mathcal{D}}_{F_A,F_B,F_C}]{} & \mathrm{Hom}_{\mathcal{D}}(F_A, F_C) \end{array}$$
commutes, i.e. for each composable pair (g, f) of morphisms of $\mathcal{C}$, we have
$$F_{g\circ f} = F_g \circ F_f.$$

2. *Preservation of Identities.* For each $A \in \mathrm{Obj}(\mathcal{C})$, the diagram



commutes, i.e. we have

$$F_{\mathrm{id}_A} = \mathrm{id}_{F_A}.$$

[1] *Further Terminology:* Also called a **covariant functor**.

[2] *Einstein Notation:* Given functors $F \colon \mathcal{C} \longrightarrow \mathcal{D}$ and $G \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{D}$, we write F_A for $F(A)$ (resp. G^A for $G(A)$) and F_f for $F(f)$ (resp. G^f for $G(f)$).

Example 2.1.2 ▶ Identity Functors

The **identity functor** of a category $\mathcal{C}$ is the functor $\mathrm{id}_{\mathcal{C}} \colon \mathcal{C} \longrightarrow \mathcal{C}$ where

1. *Action on Objects.* For each $A \in \mathrm{Obj}(\mathcal{C})$, we have

$$\mathrm{id}_{\mathcal{C}}(A) \stackrel{\mathrm{def}}{=} A;$$

2. *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms map

$$(\mathrm{id}_{\mathcal{C}})_{A,B} \colon \mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \underbrace{\mathrm{Hom}_{\mathcal{C}}(\mathrm{id}_{\mathcal{C}}(A), \mathrm{id}_{\mathcal{C}}(B))}_{\stackrel{\mathrm{def}}{=} \mathrm{Hom}_{\mathcal{C}}(A,B)}$$

of $\mathrm{id}_{\mathcal{C}}$ at (A, B) is defined by

$$(\mathrm{id}_{\mathcal{C}})_{A,B} \stackrel{\mathrm{def}}{=} \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}.$$

Proof 2.1.3 ▶ Proof of Example 2.1.2

Preservation of Identities

We have $\mathrm{id}_{\mathcal{C}}(\mathrm{id}_A) \stackrel{\mathrm{def}}{=} \mathrm{id}_A$ for each $A \in \mathrm{Obj}(\mathcal{C})$ by definition.

Preservation of Compositions

For each composable pair $A \xrightarrow{f} B \xrightarrow{g} B$ of morphisms of $\mathcal{C}$, we have

$$\begin{aligned}\mathrm{id}_{\mathcal{C}}(g \circ f) &\stackrel{\text{def}}{=} g \circ f \\ &\stackrel{\text{def}}{=} \mathrm{id}_{\mathcal{C}}(g) \circ \mathrm{id}_{\mathcal{C}}(f).\end{aligned}$$

This finishes the proof.

Proposition-Definition 2.1.4 ▶ Composition of Functors

The **composition** of two functors $F: \mathcal{C} \longrightarrow \mathcal{D}$ and $G: \mathcal{D} \longrightarrow \mathcal{E}$ is the functor $G \circ F$ where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathcal{C})$, we have

$$(G \circ F)_A \stackrel{\text{def}}{=} G_{F_A};$$

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms map

$$(G \circ F)_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{E}}\left(G_{F_A}, G_{F_B}\right)$$

  of $G \circ F$ at (A, B) is defined by

$$(G \circ F)_f \stackrel{\text{def}}{=} G_{F_f}.$$

Proof 2.1.5 ▶ Proof of Proposition-Definition 2.1.4

Preservation of Identities

For each $A \in \mathrm{Obj}(\mathcal{C})$, we have

$$\begin{aligned}G_{F_{\mathrm{id}_A}} &= G_{\mathrm{id}_{F_A}} && \text{(by the functoriality of } F\text{)} \\ &= \mathrm{id}_{G_{F_A}}. && \text{(by the functoriality of } G\text{)}\end{aligned}$$

Preservation of Composition

For each composable pair (g, f) of morphisms of $\mathcal{C}$, we have

$$\begin{aligned}G_{F_{g \circ f}} &= G_{F_g \circ F_f} && \text{(by the functoriality of } F\text{)} \\ &= G_{F_g} \circ G_{F_f}. && \text{(by the functoriality of } G\text{)}\end{aligned}$$

This finishes the proof. ∎

2.1.2 Conditions on Functors

Definition 2.1.6 ▶ Conditions on Functors

A functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is

1. **Faithful** if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms map

$$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)$$

   of F at (A, B) is injective.

2. **Full** if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms map

$$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)$$

   of F at (A, B) is surjective.

3. **Fully faithful** if F is full and faithful, i.e. if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms map

$$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)$$

   of F at (A, B) is bijective.

4. **Conservative** if whenever F_f is an isomorphism in $\mathcal{D}$, so is f in $\mathcal{C}$.[1]

5. **Essentially surjective** if, for each $D \in \mathrm{Obj}(\mathcal{D})$, there exists some object A of $\mathcal{C}$ such that $F_A \cong D$.

[1] Since functors preserve isomorphisms, we see that F is conservative iff, for each $f \in \mathrm{Mor}(\mathcal{C})$, we have

$$(f \text{ is an isomorphism}) \iff (F_f \text{ is an isomorphism}).$$

Proposition 2.1.7 ▶ Fully Faithful Functors Are Conservative

Every fully faithful functor is conservative.

Proof 2.1.8 ▶ Proof of Proposition 2.1.7

Let $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a fully faithful functor, $f\colon A \longrightarrow B$ be a morphism of $\mathcal{C}$, and suppose that F_f is an isomorphism. Then we have

$$\begin{aligned} F_{\mathrm{id}_B} &= \mathrm{id}_{F_B} \\ &= F_f \circ F_f^{-1} \\ &= F_{f \circ f^{-1}}. \end{aligned}$$

Similarly, $F_{\mathrm{id}_A} = F_{f^{-1} \circ f}$. As F is fully faithful, we have

$$\begin{aligned} f \circ f^{-1} &= \mathrm{id}_B, \\ f^{-1} \circ f &= \mathrm{id}_A. \end{aligned}$$

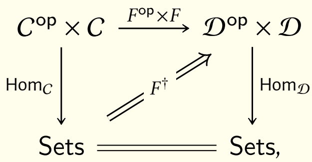
Hence f is an isomorphism and F is conservative.

2.1.3 The Natural Transformation Associated to a Functor

Proposition 2.1.9 ▶ The Natural Transformation Associated to a Functor

Every functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ defines a natural transformation

$$F^{\dagger}\colon \mathrm{Hom}_{\mathcal{C}} \Longrightarrow \mathrm{Hom}_{\mathcal{D}} \circ (F^{\mathrm{op}} \times F),$$



called the **natural transformation associated to** F, consisting of the collection

$$\left\{F^{\dagger}_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A,B) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F_A,F_B)\right\}_{(A,B)\in \mathrm{Obj}(\mathcal{C}^{\mathrm{op}}\times\mathcal{C})}$$

with

$$F^{\dagger}_{A,B} \overset{\mathrm{def}}{=} F_{A,B}.$$

Proof 2.1.10 ▶ Proof of Proposition 2.1.9

The naturality condition for $F^\dagger$ is the requirement that for each morphism

$$(\phi, \psi)\colon (X, Y) \longrightarrow (A, B)$$

of $\mathcal{C}^{\mathsf{op}} \times \mathcal{C}$, the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(X, Y) & \xrightarrow{\overbrace{(\phi,\psi)}^{\stackrel{\text{def}}{=}\psi\circ(-)\circ\phi}} & \mathrm{Hom}_{\mathcal{C}}(A, B) \\ {\scriptstyle F_{X,Y}}\downarrow & & \downarrow{\scriptstyle F_{A,B}} \\ \mathrm{Hom}_{\mathcal{D}}(F_X, F_Y) & \xrightarrow[\underbrace{[F^{\mathsf{op}}\times F](\phi,\psi)}_{\stackrel{\text{def}}{=}F_\psi\circ(-)\circ F_\phi}]{} & \mathrm{Hom}_{\mathcal{D}}(F_A, F_B), \end{array}$$

acting on elements as

$$\begin{array}{ccc} f & \longmapsto & \psi \circ f \circ \phi \\ \downarrow & & \downarrow \\ F_f & \longmapsto & F_\psi \circ F_f \circ F_\phi = F_{\psi\circ f\circ\phi} \end{array}$$

commutes, which follows from the functoriality of F. ∎

2.2 Natural Transformations

2.2.1 Foundations

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.

Definition 2.2.1 ▶ Transformations

A **transformation**[1,2] $\alpha\colon F \overset{\text{unnat}}{\Longrightarrow} G$ **from** F **to** G is a collection

$$\{\alpha_A\colon F_A \longrightarrow G_A\}_{A\in\mathrm{Obj}(\mathcal{C})}$$

of morphisms of $\mathcal{D}$.

[1] *Further Terminology:* Also called an **unnatural transformation** for emphasis.
[2] *Further Notation:* We write $\mathrm{UnNat}(F, G)$ for the set of unnatural transformations from F to G.

Definition 2.2.2 ▶ Natural Transformations

A **natural transformation**[1] $\alpha\colon F \Longrightarrow G$ **from** F **to** G is a transformation

$$\{\alpha_A\colon F_A \longrightarrow G_A\}_{A\in\mathrm{Obj}(\mathcal{C})}$$

from F to G such that, for each morphism $f\colon A \longrightarrow B$ of $\mathcal{C}$, the diagram

$$\begin{array}{ccc} F_A & \xrightarrow{F_f} & F_B \\ {\scriptstyle \alpha_A}\downarrow & & \downarrow{\scriptstyle \alpha_B} \\ G_A & \xrightarrow[G_f]{} & G_B \end{array}$$

commutes.[2,3]

[1] Pictured in diagrams as

$$\mathcal{C} \underset{G}{\overset{F}{\Downarrow {\scriptstyle\alpha}}} \mathcal{D}.$$

[2] *Further Terminology:* The morphism $\alpha_A\colon F_A \longrightarrow G_A$ is called the **component of** α **at** A.

[3] *Further Notation:* We write $\mathrm{Nat}(F, G)$ for the set of natural transformations from F to G.

Example 2.2.3 ▶ Identity Natural Transformations

The **identity natural transformation** $\mathrm{id}_F\colon F \Longrightarrow F$ **of** F is the natural transformation consisting of the collection

$$\{\mathrm{id}_{F_A}\colon F_A \longrightarrow F_A\}_{A\in\mathrm{Obj}(\mathcal{C})}.$$

Proof 2.2.4 ▶ Proof of Example 2.2.3

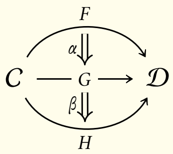
The naturality condition for id_F is the requirement that, for each morphism $f\colon A \longrightarrow B$ of $\mathcal{C}$, the diagram

$$\begin{array}{ccc} F_A & \xrightarrow{F_f} & F_B \\ {\scriptstyle \mathrm{id}_{F_A}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{F_B}} \\ F_A & \xrightarrow[F_f]{} & F_B \end{array}$$

commutes, which follows from unitality of the composition of $\mathcal{C}$. ∎

Definition 2.2.5 ▶ Vertical Composition of Natural Transformations

The **vertical composition** of two natural transformations $\alpha\colon F \Longrightarrow G$ and $\beta\colon G \Longrightarrow H$ as in the diagram



is the natural transformation $\beta \circ \alpha\colon F \Longrightarrow H$ consisting of the collection

$$\left\{(\beta \circ \alpha)_A\colon F_A \longrightarrow H_A\right\}_{A \in \mathrm{Obj}(\mathcal{C})}$$

with

$$(\beta \circ \alpha)_A \overset{\text{def}}{=} \beta_A \circ \alpha_A$$

for each $A \in \mathrm{Obj}(\mathcal{C})$.

Proof 2.2.6 ▶ Proof of Definition 2.2.5

The naturality condition for $\beta \circ \alpha$ is the requirement that the boundary of the diagram

$$\begin{array}{ccc} F_A & \xrightarrow{F_f} & F_B \\ {\scriptstyle \alpha_A}\downarrow & (1) & \downarrow{\scriptstyle \alpha_B} \\ G_A & \xrightarrow{G_f} & G_B \\ {\scriptstyle \beta_A}\downarrow & (2) & \downarrow{\scriptstyle \beta_B} \\ H_A & \xrightarrow[H_f]{} & H_B \end{array}$$

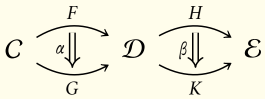
commutes. Since

1. Subdiagram (1) commutes by the naturality of α;
2. Subdiagram (2) commutes by the naturality of β;

so does the boundary diagram. Hence $\beta \circ \alpha$ is a natural transformation. ◻

Definition 2.2.7 ▶ Horizontal Composition of Natural Transformations

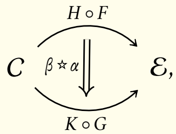
The **horizontal composition**[1] of two natural transformations $\alpha\colon F \Longrightarrow G$ and $\beta\colon H \Longrightarrow K$ as in the diagram



of α and β is the natural transformation

$$\beta \star \alpha\colon (H \circ F) \Longrightarrow (K \circ G),$$

as in the diagram



consisting of the collection

$$\left\{(\beta \star \alpha)_A\colon H_{F_A} \longrightarrow K_{G_A}\right\}_{A \in \mathrm{Obj}(\mathcal{C})},$$

of morphisms of $\mathcal{E}$ with

$$\begin{aligned}(\beta \star \alpha)_A &\overset{\text{def}}{=} \beta_{G_A} \circ H_{\alpha_A} \\ &= K_{\alpha_A} \circ \beta_{F_A},\end{aligned} \qquad \begin{array}{ccc} H_{F_A} & \xrightarrow{H_{\alpha_A}} & H_{G_A} \\ {\scriptstyle \beta_{F_A}}\downarrow & & \downarrow{\scriptstyle \beta_{G_A}} \\ K_{F_A} & \xrightarrow[K_{\alpha_A}]{} & K_{G_A}. \end{array}$$

[1] *Further Terminology*: Also called the **Godement product** of α and β.

PROOF 2.2.8 ▶ PROOF OF DEFINITION 2.2.7

The naturality condition for $\beta \star \alpha$ is the requirement that the boundary of the diagram

$$\begin{array}{ccc}
H_{F_A} & \xrightarrow{H_{F_f}} & H_{F_B} \\
{\scriptstyle H_{\alpha_A}}\big\downarrow & (1) & \big\downarrow{\scriptstyle H_{\alpha_B}} \\
H_{G_A} & \xrightarrow{H_{G_f}} & H_{G_B} \\
{\scriptstyle \beta_{G_A}}\big\downarrow & (2) & \big\downarrow{\scriptstyle \beta_{G_B}} \\
K_{G_A} & \xrightarrow[K_{G_f}]{} & K_{G_B}
\end{array}$$

commutes. Since

1. Subdiagram (1) commutes by the naturality of α;
2. Subdiagram (2) commutes by the naturality of β;

so does the boundary diagram. Hence $\beta \circ \alpha$ is a natural transformation.[1]

[1] *Reference:* [Bor94b, Proposition 1.3.4].

2.2.2 Properties of Natural Transformations

PROPOSITION 2.2.9 ▶ NATURAL TRANSFORMATIONS AS HOMOTOPIES

[1]Let $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors. The following data are equivalent:

1. A natural transformation $\alpha\colon F \Longrightarrow G$.

2. A functor $[\alpha]\colon \mathcal{C} \longrightarrow \mathcal{D}^{I}$ filling the diagram

$$\begin{array}{ccc} & & \mathcal{D} \\ & \overset{F}{\nearrow} & \uparrow \text{ev}_0 \\ \mathcal{C} & \dashrightarrow & \mathcal{D}^{I} \\ & \underset{G}{\searrow} & \downarrow \text{ev}_1 \\ & & \mathcal{D} \end{array} \tag{2.2.1}$$

3. A functor $[\alpha]\colon \mathcal{C} \times I \longrightarrow \mathcal{D}$ filling the diagram

$$\begin{array}{ccc} \mathcal{C} & & \\ \text{ev}_0 \uparrow & \overset{F}{\searrow} & \\ \mathcal{C} \times I & \dashrightarrow & \mathcal{D} \\ \text{ev}_1 \downarrow & \underset{G}{\nearrow} & \\ \mathcal{C} & & \end{array} \tag{2.2.2}$$

[1]Taken from [MO MO64365].

Proof 2.2.10 ▶ Proof of Proposition 2.2.9

Item 1 $\iff$ Item 2

By **??**, we may identify $\mathcal{D}^{I}$ with $\text{Arr}(\mathcal{D})$. Given a natural transformation $\alpha\colon F \Longrightarrow$

G, we have a functor

$$
\begin{aligned}
[\alpha]\colon \mathcal{C} &\longrightarrow \mathcal{D}^{|} \\
A &\longmapsto \alpha_A \\
(f\colon A \longrightarrow B) &\longmapsto \left(\begin{array}{ccc} F_A & \xrightarrow{F_f} & F_B \\ {\scriptstyle \alpha_A}\big\downarrow & & \big\downarrow{\scriptstyle \alpha_B} \\ G_A & \xrightarrow[G_f]{} & G_B \end{array}\right)
\end{aligned}
$$

making Diagram (2.2.1) commute. Conversely, every such functor gives rise to a natural transformation from F to G.

Item 2 $\Longleftrightarrow$ Item 3

This follows from **??** of Proposition 2.3.2.

Proposition 2.2.11 ▶ Properties of Composition of Natural Transformations

Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be categories.

1. *Vertical Composition Is Strictly Associative and Unital.* Let $F, G, H, K\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors and

   $$F \overset{\alpha}{\Longrightarrow} G \overset{\beta}{\Longrightarrow} H \overset{\gamma}{\Longrightarrow} K$$

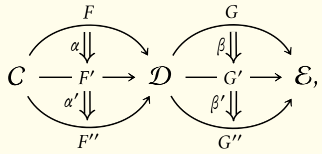
   be natural transformations. Then

   $$
   \begin{aligned}
   \mathrm{id}_G \circ \alpha &= \alpha, \\
   \alpha \circ \mathrm{id}_F &= \alpha, \\
   (\gamma \circ \beta) \circ \alpha &= \gamma \circ (\beta \circ \alpha).
   \end{aligned}
   $$

2. *Horizontal Composition of Natural Transformations Preserves Identities.* Let $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ and $G\colon \mathcal{D} \longrightarrow \mathcal{E}$ be functors. We have

   $$\mathrm{id}_G \star \mathrm{id}_F = \mathrm{id}_{G \circ F}.$$

3. *Middle Four Exchange.* Given natural transformations α, α', β, and β' as in the diagram



we have

$$(\beta' \star \alpha') \circ (\beta \star \alpha) = (\beta' \circ \beta) \star (\alpha' \circ \alpha).$$

PROOF 2.2.12 ▶ PROOF OF PROPOSITION 2.2.11

Item 1: Vertical Composition Is Strictly Associative and Unital

This follows from the fact that these identities hold at each component. In detail, given $A \in \mathrm{Obj}(\mathcal{C})$, we have

$$\begin{aligned}(\mathrm{id}_G \circ \alpha)_A &= \mathrm{id}_G \circ \alpha_A = \alpha_A,\\ (\alpha \circ \mathrm{id}_F)_A &= \alpha_A \circ \mathrm{id}_F = \alpha_A.\end{aligned}$$

Similarly, we have

$$\begin{aligned}((\gamma \circ \beta) \circ \alpha)_A &= (\gamma_A \circ \beta_A) \circ \alpha_A\\ &= \gamma_A \circ (\beta_A \circ \alpha_A)\\ &= (\gamma \circ (\beta \circ \alpha))_A.\end{aligned}$$

Item 2: Horizontal Composition of Natural Transformations Preserves Identities

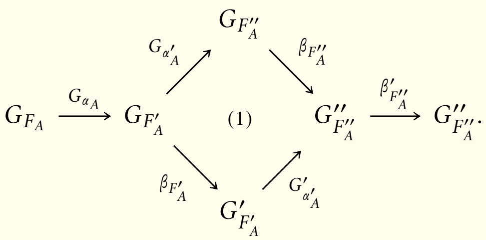
For each $A \in \mathrm{Obj}(\mathcal{C})$, we have

$$\begin{aligned}(\mathrm{id}_G \star \mathrm{id}_F)_A &\overset{\text{def}}{=} (\mathrm{id}_G)_{F_A} \circ G_{(\mathrm{id}_F)_A}\\ &\overset{\text{def}}{=} \mathrm{id}_{G_{F_A}} \circ G_{\mathrm{id}_{F_A}}\\ &= \mathrm{id}_{G_{F_A}} \circ \mathrm{id}_{G_{F_A}}\\ &= \mathrm{id}_{G_{F_A}}\\ &\overset{\text{def}}{=} (\mathrm{id}_{G \circ F})_A.\end{aligned}$$

Hence $\mathrm{id}_G \star \mathrm{id}_F = \mathrm{id}_{G \circ F}$.

Item 3: Middle Four Exchange

Let $A \in \mathrm{Obj}(C)$ and consider the diagram



The top composition is $((\beta' \circ \beta) \star (\alpha' \circ \alpha))_A$ and the bottom composition is $((\beta' \star \alpha') \circ (\beta \star \alpha))_A$. Since Subdiagram (1) commutes, they are equal.

Definition 2.2.13 ▶ Equality of Natural Transformations

Two natural transformations $\alpha, \beta \colon F \Longrightarrow G$ are **equal** if, for each $A \in \mathrm{Obj}(C)$, we have

$$\alpha_A = \beta_A.$$

2.2.3 Natural Isomorphisms

Definition 2.2.14 ▶ Natural Isomorphisms

A natural transformation $\alpha \colon F \Longrightarrow G$ between functors $F, G \colon C \longrightarrow \mathcal{D}$ between categories C and $\mathcal{D}$ is a **natural isomorphism** if there exists a natural transformation $\alpha^{-1} \colon G \Longrightarrow F$ such that

$$\alpha \circ \alpha^{-1} = \mathrm{id}_G,$$
$$\alpha^{-1} \circ \alpha = \mathrm{id}_F.$$

Proposition 2.2.15 ▶ Properties of Natural Isomorphisms

Let $\alpha \colon F \Longrightarrow G$ be a natural transformation.

1. *Characterisations.* The following conditions are equivalent:

   (a) The natural transformation α is a natural isomorphism.

(b) For each $A \in \mathsf{Obj}(\mathcal{C})$, the morphism $\alpha_A \colon F_A \longrightarrow G_A$ is an isomorphism.

Proof 2.2.16 ▶ Proof of Proposition 2.2.15

Omitted.

2.3 Categories of Categories

2.3.1 Functor Categories

Let $\mathcal{C}$ be a category and $\mathcal{D}$ be a small category.

Definition 2.3.1 ▶ Functor Categories

The **category of functors from $\mathcal{C}$ to $\mathcal{D}$**[1] is the category $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$[2] where

- *Objects*. The objects of $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ are functors from $\mathcal{C}$ to $\mathcal{D}$;
- *Morphisms*. For each $F, G \in \mathsf{Obj}(\mathsf{Fun}(\mathcal{C}, \mathcal{D}))$, we have

$$\mathsf{Hom}_{\mathsf{Fun}(\mathcal{C},\mathcal{D})}(F, G) \stackrel{\text{def}}{=} \mathsf{Nat}(F, G);$$

- *Identities*. For each $F \in \mathsf{Obj}(\mathsf{Fun}(\mathcal{C}, \mathcal{D}))$, the unit map

$$\mathbb{1}^{\mathsf{Fun}(\mathcal{C},\mathcal{D})}_F \colon \mathrm{pt} \longrightarrow \mathsf{Nat}(F, F)$$

  of $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ at F is given by

$$\mathrm{id}^{\mathsf{Fun}(\mathcal{C},\mathcal{D})}_F \stackrel{\text{def}}{=} \mathrm{id}_F,$$

  where $\mathrm{id}_F \colon F \Longrightarrow F$ is the identity natural transformation of F of Example 2.2.3;

- *Composition*. For each $F, G, H \in \mathsf{Obj}(\mathsf{Fun}(\mathcal{C}, \mathcal{D}))$, the composition map

$$\circ^{\mathsf{Fun}(\mathcal{C},\mathcal{D})}_{F,G,H} \colon \mathsf{Nat}(G, H) \times \mathsf{Nat}(F, G) \longrightarrow \mathsf{Nat}(F, H)$$

  of $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ at (F, G, H) is given by

$$\beta \circ^{\mathsf{Fun}(\mathcal{C},\mathcal{D})}_{F,G,H} \alpha \stackrel{\text{def}}{=} \beta \circ \alpha,$$

  where $\beta \circ \alpha$ is the vertical composition of α and β of Definition 2.2.5.

[1] Or the **functor category** $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$.
[2] *Further Notation:* Also written $\mathcal{D}^{\mathcal{C}}$ and $[\mathcal{C}, \mathcal{D}]$.

Proposition 2.3.2 ▶ Properties of Functor Categories

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a functor.

1. *Functoriality.* The assignments $\mathcal{C}, \mathcal{D}, (\mathcal{C}, \mathcal{D}) \mapsto \mathsf{Fun}(\mathcal{C}, \mathcal{D})$ define functors

$$\begin{aligned}\mathsf{Fun}(\mathcal{C}, -_2)&\colon \mathsf{Cats} \longrightarrow \mathsf{Cats},\\ \mathsf{Fun}(-_1, \mathcal{D})&\colon \mathsf{Cats}^{\mathsf{op}} \longrightarrow \mathsf{Cats},\\ \mathsf{Fun}(-_1, -_2)&\colon \mathsf{Cats}^{\mathsf{op}} \times \mathsf{Cats} \longrightarrow \mathsf{Cats}.\end{aligned}$$

2. *2-Functoriality.* The assignments $\mathcal{C}, \mathcal{D}, (\mathcal{C}, \mathcal{D}) \mapsto \mathsf{Fun}(\mathcal{C}, \mathcal{D})$ define 2-functors

$$\begin{aligned}\mathsf{Fun}(\mathcal{C}, -_2)&\colon \mathsf{Cats}_2 \longrightarrow \mathsf{Cats}_2,\\ \mathsf{Fun}(-_1, \mathcal{D})&\colon \mathsf{Cats}_2^{\mathsf{op}} \longrightarrow \mathsf{Cats}_2,\\ \mathsf{Fun}(-_1, -_2)&\colon \mathsf{Cats}_2^{\mathsf{op}} \times \mathsf{Cats}_2 \longrightarrow \mathsf{Cats}_2.\end{aligned}$$

3. *2-Adjointness.* We have 2-adjunctions

$$(\mathcal{C} \times - \dashv \mathsf{Fun}(\mathcal{C}, -))\colon \quad \mathsf{Cats}_2 \underset{\mathsf{Fun}(\mathcal{C},-)}{\overset{\mathcal{C}\times -}{\rightleftarrows}} \mathsf{Cats}_2, \quad (\perp_2)$$

$$(- \times \mathcal{D} \dashv \mathsf{Fun}(\mathcal{D}, -))\colon \quad \mathsf{Cats}_2 \underset{\mathsf{Fun}(\mathcal{D},-)}{\overset{-\times\mathcal{D}}{\rightleftarrows}} \mathsf{Cats}_2, \quad (\perp_2)$$

witnessed by isomorphisms of categories

$$\begin{aligned}\mathsf{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) &\cong \mathsf{Fun}(\mathcal{D}, \mathsf{Fun}(\mathcal{C}, \mathcal{E})),\\ \mathsf{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) &\cong \mathsf{Fun}(\mathcal{C}, \mathsf{Fun}(\mathcal{D}, \mathcal{E})),\end{aligned}$$

natural in $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats}_2)$.

4. *Adjointness.* We have adjunctions

$$(\mathcal{C} \times - \dashv \mathsf{Fun}(\mathcal{C}, -))\colon \quad \mathsf{Cats} \underset{\mathsf{Fun}(\mathcal{C},-)}{\overset{\mathcal{C}\times -}{\rightleftarrows}} \mathsf{Cats}, \quad (\perp)$$

$$(- \times \mathcal{D} \dashv \mathsf{Fun}(\mathcal{D}, -))\colon \quad \mathsf{Cats} \underset{\mathsf{Fun}(\mathcal{D},-)}{\overset{-\times\mathcal{D}}{\rightleftarrows}} \mathsf{Cats}, \quad (\perp)$$

witnessed by bijections of sets

$$\mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{D}, \mathrm{Fun}(\mathcal{C}, \mathcal{E})),$$
$$\mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathrm{Fun}(\mathcal{D}, \mathcal{E})),$$

natural in $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats})$.

5. *Trivial Functor Categories.* We have a canonical isomorphism of categories

$$\mathrm{Fun}(\mathrm{pt}, \mathcal{C}) \cong \mathcal{C},$$

natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$.

6. *Characterisations of Fully Faithfulness.* The following conditions are equivalent:
   (a) The functor $F \colon \mathcal{C} \longrightarrow \mathcal{D}$ is fully faithful.
   (b) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the functor

   $$F^{*} \colon \mathrm{Fun}(\mathcal{D}, \mathcal{X}) \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{X})$$

   is fully faithful.
   (c) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the functor

   $$F_{*} \colon \mathrm{Fun}(\mathcal{X}, \mathcal{C}) \longrightarrow \mathrm{Fun}(\mathcal{X}, \mathcal{D})$$

   is fully faithful.

7. *Objectwise Computation of Co/Limits.* Let

$$D \colon \mathcal{I} \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

be a diagram in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. We have isomorphisms

$$\lim(D)_A \cong \lim_{i \in \mathcal{I}}(D_i(A)),$$
$$\mathrm{colim}(D)_A \cong \mathrm{colim}_{i \in \mathcal{I}}(D_i(A)),$$

naturally in $A \in \mathrm{Obj}(\mathcal{C})$.

8. *Bicompleteness.* If $\mathcal{E}$ is co/complete, then so is $\mathrm{Fun}(\mathcal{C}, \mathcal{E})$.

9. *Abelianness.* If $\mathcal{E}$ is abelian, then so is $\mathrm{Fun}(\mathcal{C}, \mathcal{E})$.

10. *Monomorphisms and Epimorphisms.* Let $\alpha\colon F \Longrightarrow G$ be a morphism of $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$. The following conditions are equivalent:

   (a) The natural transformation

   $$\alpha\colon F \Longrightarrow G$$

   is a monomorphism (resp. epimorphism) in $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$.

   (b) For each $A \in \mathrm{Obj}(\mathcal{C})$, the morphism

   $$\alpha_A\colon F_A \longrightarrow G_A$$

   is a monomorphism (resp. epimorphism) in $\mathcal{D}$.

Proof 2.3.3 ▶ Proof of Proposition 2.3.2

Item 1: Functoriality

Omitted.

Item 2: 2-Functoriality

Omitted.

Item 3: 2-Adjointness

Omitted.

Item 4: Adjointness

Omitted.

Item 5: Trivial Functor Categories

Omitted.

Item 6: Characterisations of Fully Faithfulness

See [Low15, Propositions A.I.5].

Item 7: Objectwise Computation of Co/Limits

Omitted.

Item 8: Bicompleteness

This follows from **??**.

Item 9: Abelianness

Omitted.

Item 10: Monomorphisms and Epimorphisms

Omitted.

2.3.2 The Category of Categories and Functors

Definition 2.3.4 ▶ The Category of Categories and Functors

The **category of (small) categories and functors** is the category Cats where

- *Objects.* The objects of Cats are small categories;
- *Morphisms.* For each $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$, we have

$$\mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{D}) \stackrel{\text{def}}{=} \mathrm{Obj}(\mathsf{Fun}(\mathcal{C}, \mathcal{D}));$$

- *Identities.* For each $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$, the unit map

$$\mathbb{1}^{\mathsf{Cats}}_{\mathcal{C}} \colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{C})$$

  of Cats at $\mathcal{C}$ is defined by

$$\mathrm{id}^{\mathsf{Cats}}_{\mathcal{C}} \stackrel{\text{def}}{=} \mathrm{id}_{\mathcal{C}},$$

  where $\mathrm{id}_{\mathcal{C}} \colon \mathcal{C} \longrightarrow \mathcal{C}$ is the identity functor of $\mathcal{C}$ of Example 2.1.2;
- *Composition.* For each $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats})$, the composition map

$$\circ^{\mathsf{Cats}}_{\mathcal{C},\mathcal{D},\mathcal{E}} \colon \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{D}, \mathcal{E}) \times \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{D}) \longrightarrow \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{E})$$

  of Cats at $(\mathcal{C}, \mathcal{D}, \mathcal{E})$ is given by

$$G \circ^{\mathsf{Cats}}_{\mathcal{C},\mathcal{D},\mathcal{E}} F \stackrel{\text{def}}{=} G \circ F,$$

  where $G \circ F \colon \mathcal{C} \longrightarrow \mathcal{E}$ is the composition of F and G of Proposition-Definition 2.1.4.

Proposition 2.3.5 ▶ Properties of the Category Cats

Let $\mathcal{C}$ be a category.

1. *Co/Completeness.* The category Cats is complete and cocomplete.

2. *Cartesian Monoidal Structure.* The quadruple $(\mathsf{Cats}, \times, \mathsf{pt}, \mathsf{Fun})$ is a Cartesian closed monoidal category.

PROOF 2.3.6 ▶ PROOF OF PROPOSITION 2.3.5

Item 1: Co/Completeness

See [Lor21, Proposition A.4.20].

Item 2: Cartesian Monoidal Structure

Omitted.

2.3.3 The 2-Category of Categories, Functors, and Natural Transformations

DEFINITION 2.3.7 ▶ THE 2-CATEGORY OF CATEGORIES

The 2**-category of (small) categories, functors, and natural transformations** is the 2-category Cats_2 where

- *Objects.* The objects of Cats_2 are small categories;
- Hom-*Categories.* For each $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats}_2)$, we have

$$\mathrm{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D}) \stackrel{\text{def}}{=} \mathsf{Fun}(\mathcal{C}, \mathcal{D});$$

- *Identities.* For each $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats}_2)$, the unit functor

$$\mathbb{1}^{\mathsf{Cats}_2}_{\mathcal{C}} \colon \mathsf{pt} \longrightarrow \mathsf{Fun}(\mathcal{C}, \mathcal{C})$$

  of Cats_2 at $\mathcal{C}$ is the functor picking the identity functor $\mathrm{id}_{\mathcal{C}} \colon \mathcal{C} \longrightarrow \mathcal{C}$ of $\mathcal{C}$;
- *Composition.* For each $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats}_2)$, the composition bifunctor

$$\circ^{\mathsf{Cats}_2}_{\mathcal{C},\mathcal{D},\mathcal{E}} \colon \mathrm{Hom}_{\mathsf{Cats}_2}(\mathcal{D}, \mathcal{E}) \times \mathrm{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D}) \longrightarrow \mathrm{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{E})$$

  of Cats_2 at $(\mathcal{C}, \mathcal{D}, \mathcal{E})$ is the functor where
  - *Action on Objects.* For each object $(G, F) \in \mathrm{Obj}(\mathrm{Hom}_{\mathsf{Cats}_2}(\mathcal{D}, \mathcal{E}) \times \mathrm{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D}))$, we have

$$\circ^{\mathsf{Cats}_2}_{\mathcal{C},\mathcal{D},\mathcal{E}}(G, F) \stackrel{\text{def}}{=} G \circ F;$$

- *Action on Morphisms.* For each morphism $(\beta, \alpha)\colon (K, H) \Longrightarrow (G, F)$ of $\mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{D}, \mathcal{E}) \times \mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D})$, we have

$$\circ^{\mathsf{Cats}_2}_{\mathcal{C},\mathcal{D},\mathcal{E}}(\beta, \alpha) \overset{\text{def}}{=} \beta \star \alpha,$$

where $\beta \star \alpha$ is the horizontal composition of α and β of Definition 2.2.7.

2.3.4 The Category of Groupoids

Definition 2.3.8 ▶ The Category of Small Groupoids

The **category of small groupoids** is the full subcategory Grpd of Cats spanned by the groupoids.

2.3.5 The 2-Category of Groupoids

Definition 2.3.9 ▶ The 2-Category of Small Groupoids

The **2-category of small groupoids** is the full sub-2-category Grpd_2 of Cats_2 spanned by the groupoids.

2.4 Equivalences of Categories

Definition 2.4.1 ▶ Equivalences of Categories

An **equivalence of categories** consists of a pair of functors

$$F\colon \mathcal{C} \rightleftarrows \mathcal{D} :G$$

together with natural isomorphisms $F \circ G \cong \mathrm{id}_{\mathcal{D}}$ and $G \circ F \cong \mathrm{id}_{\mathcal{C}}$.[1]

[1] In this situation, some authors call the functor G a **quasi-inverse** to F.

Proposition 2.4.2 ▶ Properties of Equivalences of Categories

Let $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a functor.

1. *Characterisations.* If $\mathcal{C}$ and $\mathcal{D}$ are small[1], then the following conditions are equivalent:[2]

(a) The functor F is an equivalence of categories.

(b) The functor F is fully faithful and essentially surjective.

(c) The induced functor $F|_{\mathsf{Sk}(\mathcal{C})} \colon \mathsf{Sk}(\mathcal{C}) \longrightarrow \mathsf{Sk}(\mathcal{D})$ is an *isomorphism* of categories.

2. *Two-Out-of-Three.* Let

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{G \circ F} & \mathcal{E} \\ & {\scriptstyle F} \searrow \quad \nearrow {\scriptstyle G} & \\ & \mathcal{D} & \end{array} \tag{2.4.1}$$

be a diagram in Cats. If two out of the three functors among F, G, and $G \circ F$ in Diagram (2.4.1) are equivalences of categories, then so is the third.

3. *Stability Under Composition.* Let

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D} \underset{G'}{\overset{F'}{\rightleftarrows}} \mathcal{E}$$

be a diagram in Cats. If (F, G) and (F', G') are equivalences of categories, then so is their composite $(F' \circ F, G' \circ G)$.

4. *Equivalences vs. Adjoint Equivalences.* Every equivalence of categories can be promoted to an adjoint equivalence.[3]

[1] Otherwise there will be size issues here. One can also work with large categories and universes, or require F to be *constructively* essentially surjective; see [MSE 1465107].

[2] In ZFC, the equivalence between Item (a) and Item (b) is equivalent to the axiom of choice; see [MO 119454].

In Univalent Foundations, this is true without requiring neither the axiom of choice nor the law of the excluded middle.

[3] More precisely, we can promote an equivalence of categories (F, G, η, ϵ) to adjoint equivalences (F, G, η', ϵ) and (F, G, η, ϵ').

PROOF 2.4.3 ▶ PROOF OF PROPOSITION 2.4.2

Item 1: Characterisations

We claim that Items (a) to (c) are indeed equivalent:

1. *Item (a)* $\implies$ *Item (b)*. Clear.

2. *Item (b)* $\implies$ *Item (a)*. Since F is essentially surjective and $\mathcal{C}$ and $\mathcal{D}$ are

small, we can choose, using the axiom of choice, for each $B \in \mathrm{Obj}(\mathcal{D})$, an object j_B of $\mathcal{C}$ and an isomorphism $i_B : B \longrightarrow F_{j_B}$ of $\mathcal{D}$.

Since F is fully faithful, we can extend the assignment $B \mapsto j_B$ to a *unique* functor $j : \mathcal{D} \longrightarrow \mathcal{C}$ such that the isomorphisms $i_B : B \longrightarrow F_{j_B}$ assemble into a natural isomorphism $\eta : \mathrm{id}_{\mathcal{D}} \overset{\cong}{\Longrightarrow} F \circ j$, with a similar natural isomorphism $\epsilon : \mathrm{id}_{\mathcal{C}} \overset{\cong}{\Longrightarrow} j \circ F$. Hence F is an equivalence.

3. *Item (a)* $\Longrightarrow$ *Item (c)*. This follows from **??**.

Item 2: Two-Out-of-Three

Omitted.

Item 3: Stability Under Composition

Clear.

Item 4: Equivalences vs. Adjoint Equivalences

See [Rie17, Proposition 4.4.5].

2.4.1 Isomorphisms of Categories

Definition 2.4.4 ▶ Isomorphisms of Categories

An **isomorphism of categories** is a pair of functors

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

such that $F \circ G = \mathrm{id}_{\mathcal{D}}$ and $G \circ F = \mathrm{id}_{\mathcal{C}}$.

Example 2.4.5 ▶ Equivalent But Non-Isomorphic Categories

For an example of two categories which are equivalent but non-isomorphic, see [Lor21, Example A.3.12].

Proposition 2.4.6 ▶ Properties of Isomorphisms of Categories

Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor.

1. *Characterisations.* If $\mathcal{C}$ and $\mathcal{D}$ are small, then the following conditions are equivalent:

(a) The functor F is an isomorphism of categories.

(b) The functor F is fully faithful and a bijection on objects.

Proof 2.4.7 ▶ Proof of Proposition 2.4.6

Item 1: Characterisations

Omitted, but similar to Item 1 of Proposition 2.4.2.

3 Profunctors

3.1 Foundations

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 3.1.1 ▶ Profunctors

A **profunctor**[1] $\mathfrak{p}\colon \mathcal{C} \nrightarrow \mathcal{D}$ from $\mathcal{C}$ to $\mathcal{D}$ is a functor $\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \times \mathcal{C} \longrightarrow \mathsf{Sets}$.

[1] *Further Terminology:* Also called a **distributor**, a **bimodule**, a **correspondence**, or a **relator**.

Remark 3.1.2 ▶ Equivalent Definitions of Profunctors

Equivalently, we may define a profunctor from $\mathcal{C}$ to $\mathcal{D}$ as:

1. A functor $\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \times \mathcal{C} \longrightarrow \mathsf{Sets}$;
2. A functor $\mathfrak{p}\colon \mathcal{C} \longrightarrow \mathsf{PSh}(\mathcal{D})$;
3. A functor $\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \longrightarrow \mathsf{Fun}(\mathcal{C}, \mathsf{Sets})$;
4. A cocontinuous functor $\mathfrak{p}\colon \mathsf{PSh}(\mathcal{C}) \longrightarrow \mathsf{PSh}(\mathcal{D})$;

That is, we have isomorphisms of categories

$$\begin{aligned}\mathsf{Prof}(\mathcal{C}, \mathcal{D}) &\cong \mathsf{Fun}(\mathcal{C}, \mathsf{PSh}(\mathcal{D})),\\ &\cong \mathsf{Fun}(\mathcal{D}^{\mathsf{op}}, \mathsf{CoPSh}(\mathcal{C})),\\ &\cong \mathsf{Fun}^{\mathsf{cocont}}(\mathsf{PSh}(\mathcal{C}), \mathsf{PSh}(\mathcal{D})),\end{aligned}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

Proof 3.1.3 ▶ Proof of Remark 3.1.2

We claim that Items 1 to 4 are indeed equivalent:

- The equivalence between Items 1 and 2 is an instance of currying, following from the isomorphisms of categories

$$\begin{aligned}\mathsf{Fun}(\mathcal{D}^{\mathsf{op}} \times \mathcal{C}, \mathsf{Sets}) &\cong \mathsf{Fun}(\mathcal{C}, \mathsf{Fun}(\mathcal{D}^{\mathsf{op}}, \mathsf{Sets})) && \text{(Item 3 of Proposition 2.3.2)}\\ &\overset{\text{def}}{=} \mathsf{Fun}(\mathcal{C}, \mathsf{PSh}(\mathcal{D})).\end{aligned}$$

- The equivalence between Items 1 and 3 is also an instance of currying, following from the isomorphisms of categories

$$\begin{aligned}\mathsf{Fun}(\mathcal{D}^{\mathsf{op}} \times \mathcal{C}, \mathsf{Sets}) &\cong \mathsf{Fun}(\mathcal{D}^{\mathsf{op}}, \mathsf{Fun}(\mathcal{C}, \mathsf{Sets})) && \text{(Item 3 of Proposition 2.3.2)}\\ &\overset{\text{def}}{=} \mathsf{Fun}(\mathcal{D}^{\mathsf{op}}, \mathsf{Fun}(\mathcal{C}, \mathsf{Sets})).\end{aligned}$$

- The equivalence between Items 1 and 4 follows from the universal property of the category $\mathsf{PSh}(\mathcal{C})$ of presheaves on $\mathcal{C}$ as the free cocompletion of $\mathcal{C}$ via the Yoneda embedding

$$よ \colon \mathcal{C}^{\mathsf{op}} \hookrightarrow \mathsf{PSh}(\mathcal{C})$$

  of $\mathcal{C}$ into $\mathsf{PSh}(\mathcal{C})$ (**??** of Proposition 7.3.2).

This finishes the proof. ∎

3.2 The Bicategory of Profunctors

Definition 3.2.1 ▶ The Bicategory of Profunctors

The **bicategory of profunctors** is the bicategory Prof where[1]

1. *Objects.* The objects of Prof are categories;
2. *1-Morphisms.* The 1-morphisms of Prof are profunctors;
3. *2-Morphisms.* The 2-morphisms of Prof are natural transformations between profunctors;
4. *Identities.* For each $\mathcal{C} \in \mathrm{Obj}(\mathsf{Prof})$, we have

$$\mathrm{id}^{\mathsf{Prof}}_{\mathcal{C}} \overset{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(-,-);$$

5. *Composition*. For each $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Prof})$, the composition bifunctor

$$\diamond\colon \mathsf{Prof}(\mathcal{D},\mathcal{E}) \times \mathsf{Prof}(\mathcal{C},\mathcal{D}) \longrightarrow \mathsf{Prof}(\mathcal{C},\mathcal{E})$$

is defined on objects by sending profunctors $\mathfrak{p}\colon \mathcal{C} \nrightarrow \mathcal{D}$ and $\mathfrak{q}\colon \mathcal{D} \nrightarrow \mathcal{E}$ to the profunctor $\mathfrak{q} \diamond \mathfrak{p}$ of Definition 3.3.2.

[1]The bicategory Prof admits a nice strictification to a 2-category: it is biequivalent to the sub-bicategory of Cats spanned by the presheaf categories, cocontinuous functors between them, and natural transformation between these.

Proof 3.2.2 ▶ Proof of Definition 3.2.1

See Enriched Categories, Proposition-Definition 4.1.4. ∎

3.3 Operations With Profunctors

3.3.1 The Domain and Range of a Profunctor

Definition 3.3.1 ▶ The Domain and Range of a Profunctor

Let $\mathfrak{p}\colon \mathcal{C} \nrightarrow \mathcal{D}$ be a profunctor.[1]

1. The **domain of** $\mathfrak{p}$ is the presheaf $\mathrm{dom}(\mathfrak{p})\colon \mathcal{D}^{\mathsf{op}} \longrightarrow \mathsf{Sets}$ on $\mathcal{D}$ defined by

$$\mathrm{dom}(\mathfrak{p})^{-} \stackrel{\mathrm{def}}{=} \operatorname*{colim}_{B\in\mathcal{D}}(\mathfrak{p}_B^{-}).$$

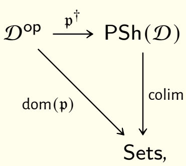
2. The **range of** $\mathfrak{p}$ is the copresheaf $\mathrm{range}(\mathfrak{p})\colon \mathcal{C} \longrightarrow \mathsf{Sets}$ on $\mathcal{C}$ defined by

$$\mathrm{range}(\mathfrak{p})_{-} \stackrel{\mathrm{def}}{=} \operatorname*{colim}_{A\in\mathcal{D}}\left(\mathfrak{p}^{A}_{-}\right).$$

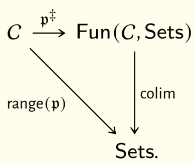
[1]In other words, the domain and range of $\mathfrak{p}$ are the functors

$$\mathrm{dom}(\mathfrak{p})\colon \mathcal{D}^{\mathsf{op}} \longrightarrow \mathsf{Sets},$$
$$\mathrm{range}(\mathfrak{p})\colon \mathcal{C} \longrightarrow \mathsf{Sets}$$

defined by



$$\mathrm{dom}(\mathfrak{p}) \stackrel{\mathrm{def}}{=} \mathrm{colim} \circ \mathfrak{p}^{\dagger},$$
$$\mathrm{range}(\mathfrak{p}) \stackrel{\mathrm{def}}{=} \mathrm{colim} \circ \mathfrak{p}^{\ddagger},$$



3.3.2 Composition of Profunctors

Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be categories and let $\mathfrak{p}\colon \mathcal{C} \nrightarrow \mathcal{D}$ and $\mathfrak{q}\colon \mathcal{D} \nrightarrow \mathcal{E}$ be profunctors.

Definition 3.3.2 ▶ Composition of Profunctors

The **composition of $\mathfrak{p}$ and $\mathfrak{q}$** is the profunctor $\mathfrak{q} \diamond \mathfrak{p}\colon \mathcal{C} \nrightarrow \mathcal{E}$ defined by[1]

$$(\mathfrak{q} \diamond \mathfrak{p})^{-_1}_{-_2} \overset{\text{def}}{=} \int^{B \in \mathcal{D}} \mathfrak{q}^{-_1}_{B} \times \mathfrak{p}^{B}_{-_2}.$$

[1] Alternatively, we may define $\mathfrak{q} \diamond \mathfrak{p}$ (using the equivalent definition of Item 2 of Remark 3.1.2) by

$$(\mathfrak{q} \diamond \mathfrak{p})^{\dagger} \overset{\text{def}}{=} \mathrm{Lan}_{\text{よ}}\left(\mathfrak{p}^{\dagger}\right) \circ \mathfrak{q}^{\dagger}, \qquad \begin{array}{ccc} & \mathcal{D} & \xrightarrow{\mathfrak{p}^{\dagger}} \mathsf{PSh}(\mathcal{C}). \\ & {\scriptstyle \text{よ}}\downarrow & \nearrow {\scriptstyle \mathrm{Lan}_{\text{よ}}(\mathfrak{p}^{\dagger})} \\ \mathcal{E} & \xrightarrow[\mathfrak{q}^{\dagger}]{} & \mathsf{PSh}(\mathcal{D}) \end{array}$$

3.3.3 Representable Profunctors

Definition 3.3.3 ▶ The Representable Profunctor Associated to a Functor

The **representable profunctor associated to a functor** $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is the profunctor $\widehat{F}^{*}\colon \mathcal{C} \nrightarrow \mathcal{D}$ defined as the adjunct of the composition

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{\text{よ}} \mathsf{PSh}(\mathcal{D})$$

under the adjunction

$$\mathsf{Fun}(\mathcal{D}^{\mathsf{op}} \times \mathcal{C}, \mathsf{Sets}) \cong \mathsf{Fun}(\mathcal{C}, \mathsf{PSh}(\mathcal{D}))$$

of Item 3 of Proposition 2.3.2.[1]

[1] That is, we have

$$\widehat{F}^{*} \overset{\text{def}}{=} \mathrm{Hom}_{\mathcal{D}}(-_1, F_{-_2}).$$

Definition 3.3.4 ▶ Representable Profunctors

A profunctor is **representable** if it is isomorphic to a representable profunctor.

Definition 3.3.5 ▶ The Corepresentable Profunctor Associated to a Functor

The **corepresentable**[1] **profunctor associated to a functor** $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is the profunctor $\widehat{F}_*\colon \mathcal{D} \nrightarrow \mathcal{C}$ defined as the adjunct of the composition

$$\mathcal{C}^{\mathsf{op}} \xrightarrow{F^{\mathsf{op}}} \mathcal{D}^{\mathsf{op}} \xrightarrow{よ} \mathsf{CoPSh}(\mathcal{D})$$

under the adjunction

$$\mathsf{Fun}(\mathcal{C}^{\mathsf{op}} \times \mathcal{D}, \mathsf{Sets}) \cong \mathsf{Fun}(\mathcal{C}^{\mathsf{op}}, \mathsf{CoPSh}(\mathcal{D}))$$

of Item 3 of Proposition 2.3.2.[2]

[1] Some authors call both $\widehat{F}^*$ and $\widehat{F}_*$ the **representable profunctors associated to** F.
[2] That is:

$$\widehat{F}_* \stackrel{\text{def}}{=} \mathsf{Hom}_{\mathcal{D}}(F_{-_1}, -_2).$$

Definition 3.3.6 ▶ Corepresentable Profunctors

A profunctor is **corepresentable** if it is isomorphic to a corepresentable profunctor.

3.3.4 Collages

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 3.3.7 ▶ The Collage of a Profunctor

The **collage** of a profunctor $\mathfrak{p}\colon \mathcal{C} \nrightarrow \mathcal{D}$ is the category $\mathsf{Coll}(\mathfrak{p})$[1] where[2]

- *Objects.* We have

$$\mathrm{Obj}(\mathsf{Coll}(\mathfrak{p})) \stackrel{\text{def}}{=} \mathrm{Obj}(\mathcal{C}) \coprod \mathrm{Obj}(\mathcal{D});$$

- *Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Coll}(\mathfrak{p}))$, we have

$$\mathsf{Hom}_{\mathsf{Coll}(\mathfrak{p})}(A, B) \stackrel{\text{def}}{=} \begin{cases} \mathsf{Hom}_{\mathcal{C}}(A, B) & \text{if } A, B \in \mathrm{Obj}(\mathcal{C}), \\ \mathsf{Hom}_{\mathcal{D}}(A, B) & \text{if } A, B \in \mathrm{Obj}(\mathcal{D}), \\ \mathfrak{p}(A, B) & \text{if } A \in \mathrm{Obj}(\mathcal{C}) \text{ and } B \in \mathrm{Obj}(\mathcal{D}), \\ \emptyset & \text{if } A \in \mathrm{Obj}(\mathcal{D}) \text{ and } B \in \mathrm{Obj}(\mathcal{C}); \end{cases}$$

- *Identities.* For each $A \in \mathrm{Obj}(\mathsf{Coll}(\mathfrak{p}))$, the unit map

$$\mathbb{1}^{\mathsf{Coll}(\mathfrak{p})}_A\colon \mathsf{pt} \longrightarrow \mathsf{Hom}_{\mathsf{Coll}(\mathfrak{p})}(A, A)$$

of $\mathsf{Coll}(\mathfrak{p})$ at A is defined by

$$\mathrm{id}_A \overset{\text{def}}{=} \begin{cases} \mathrm{id}_A^{\mathcal{C}} & \text{if } A \in \mathrm{Obj}(\mathcal{C}), \\ \mathrm{id}_A^{\mathcal{D}} & \text{if } A \in \mathrm{Obj}(\mathcal{D}); \end{cases}$$

- *Composition*. For each $A, B, C \in \mathrm{Obj}(\mathsf{Coll}(\mathfrak{p}))$, the composition map

$$\circ_{A,B,C}^{\mathsf{Coll}(\mathfrak{p})} \colon \mathrm{Hom}_{\mathsf{Coll}(\mathfrak{p})}(B, C) \times \mathrm{Hom}_{\mathsf{Coll}(\mathfrak{p})}(A, B) \longrightarrow \mathrm{Hom}_{\mathsf{Coll}(\mathfrak{p})}(A, C)$$

  of $\mathsf{Coll}(\mathfrak{p})$ at (A, B, C) is defined by[3]

$$\circ_{A,B,C}^{\mathsf{Coll}(\mathfrak{p})} \overset{\text{def}}{=} \begin{cases} \circ_{A,B,C}^{\mathcal{C}} & \text{if } A, B, C \in \mathrm{Obj}(\mathcal{C}), \\ \mathfrak{p}_C^{A,B} & \text{if } A, B \in \mathrm{Obj}(\mathcal{C}) \text{ and } C \in \mathrm{Obj}(\mathcal{D}), \\ \iota & \text{if } A, C \in \mathrm{Obj}(\mathcal{C}) \text{ and } B \in \mathrm{Obj}(\mathcal{D}), \\ \iota & \text{if } B, C \in \mathrm{Obj}(\mathcal{C}) \text{ and } A \in \mathrm{Obj}(\mathcal{D}), \\ \mathfrak{p}_{B,C}^{A} & \text{if } A \in \mathrm{Obj}(\mathcal{C}) \text{ and } B, C \in \mathrm{Obj}(\mathcal{D}), \\ \iota & \text{if } B \in \mathrm{Obj}(\mathcal{C}) \text{ and } A, C \in \mathrm{Obj}(\mathcal{D}), \\ \iota & \text{if } C \in \mathrm{Obj}(\mathcal{C}) \text{ and } A, B \in \mathrm{Obj}(\mathcal{D}), \\ \circ_{A,B,C}^{\mathcal{D}} & \text{if } A, B, C \in \mathrm{Obj}(\mathcal{D}). \end{cases}$$

---

[1] *Further Notation:* Also written $\mathcal{C} \star^{\mathfrak{p}} \mathcal{D}$, notably in [Lur09, Section 2.3.1].

[2] We also have a functor $\phi \colon \mathsf{Coll}(\mathfrak{p}) \longrightarrow \mathbb{I}$ where

- *Actions on Objects*. For each $A \in \mathrm{Obj}(\mathsf{Coll}(\mathfrak{p}))$, we have

$$\phi_A \overset{\text{def}}{=} \begin{cases} [0] & \text{if } A \in \mathrm{Obj}(\mathcal{C}), \\ [1] & \text{if } A \in \mathrm{Obj}(\mathcal{D}). \end{cases}$$

- *Actions on Morphisms*. For each $A, B \in \mathrm{Obj}(\mathsf{Coll}(\mathfrak{p}))$, the action on morphisms

$$\phi_{A,B} \colon \mathrm{Hom}_{\mathsf{Coll}(\mathfrak{p})}(A, B) \longrightarrow \mathrm{Hom}_{\mathsf{Coll}(\mathfrak{p})}(\phi_A, \phi_B)$$

  of ϕ at (A, B) is given by

$$\phi_{A,B}(f) \overset{\text{def}}{=} \begin{cases} \mathrm{id}_{[0]} & \text{if } A, B \in \mathrm{Obj}(\mathcal{C}), \\ \mathrm{id}_{[1]} & \text{if } A, B \in \mathrm{Obj}(\mathcal{D}), \\ [0] \to [1] & \text{if } A \in \mathrm{Obj}(\mathcal{C}) \text{ and } B \in \mathrm{Obj}(\mathcal{D}). \end{cases}$$

  If $A \in \mathrm{Obj}(\mathcal{D})$ and $B \in \mathrm{Obj}(\mathcal{C})$, we have $\phi_{A,B} \overset{\text{def}}{=} \mathrm{id}_{\emptyset}$.

[3] Here the maps $\mathfrak{p}_C^{A,B}$ and $\mathfrak{p}_{B,C}^{A}$ are the maps

$$\mathfrak{p}_C^{A,B} \colon \mathfrak{p}_C^B \times \mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathfrak{p}_C^A,$$
$$\mathfrak{p}_{B,C}^{A} \colon \mathrm{Hom}_{\mathcal{D}}(B, C) \times \mathfrak{p}_B^A \longrightarrow \mathfrak{p}_C^A$$

coming from the profunctor structure of $\mathfrak{p}$ and the ι's are inclusions of the empty set into the appropriate Hom sets.

Example 3.3.8 ▶ The Collage of Δ_{pt} ([Lur09, Remark 2.3.1.1])

If $\mathfrak{p}$ is the constant functor $\Delta_{\mathrm{pt}}\colon \mathcal{D}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathsf{Sets}$ with value pt, then $\mathsf{Coll}(\mathfrak{p})$ is the join $\mathcal{C} \star \mathcal{D}$ of $\mathcal{C}$ and $\mathcal{D}$ of **??**.

Proposition 3.3.9 ▶ Properties of Collages

Let $\mathfrak{p}\colon \mathcal{C} \nrightarrow \mathcal{D}$ be a profunctor.

1. *Functoriality.* The assignment $\mathfrak{p} \mapsto \mathsf{Coll}(\mathfrak{p})$ defines a functor[1]

$$\mathsf{Coll}_{\mathcal{C},\mathcal{D}}\colon \mathsf{Prof}(\mathcal{C},\mathcal{D}) \longrightarrow \mathsf{Cats}_{/I}(\mathcal{C},\mathcal{D}),$$

where

- *Action on Objects.* For each $\mathfrak{p} \in \mathrm{Obj}(\mathsf{Prof}(\mathcal{C},\mathcal{D}))$, we have

$$[\mathsf{Coll}](\mathfrak{p}) \stackrel{\text{def}}{=} \mathsf{Coll}(\mathfrak{p});$$

- *Action on Morphisms.* For each $\mathfrak{p}, \mathfrak{q} \in \mathrm{Obj}(\mathsf{Prof}(\mathcal{C},\mathcal{D}))$, the action on Hom-sets

$$\mathsf{Coll}_{\mathfrak{p},\mathfrak{q}}\colon \mathrm{Nat}(\mathfrak{p},\mathfrak{q}) \longrightarrow \mathsf{Fun}_{/I}(\mathsf{Coll}(\mathfrak{p}),\mathsf{Coll}(\mathfrak{q}))$$

of Coll at $(\mathfrak{p},\mathfrak{q})$ is the function sending a natural transformation $\alpha\colon \mathfrak{p} \Longrightarrow \mathfrak{q}$ to the functor

$$\mathsf{Coll}(\alpha)\colon \mathsf{Coll}(\mathfrak{p}) \longrightarrow \mathsf{Coll}(\mathfrak{q})$$

over I where

  - *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{Coll}(\mathfrak{p}))$, we have

$$[\mathsf{Coll}(\alpha)](X) \stackrel{\text{def}}{=} X;$$

  - *Action on Morphisms.* For each $X, Y \in \mathrm{Obj}(\mathsf{Coll}(\mathfrak{p}))$, the action on Hom-sets

$$\mathsf{Coll}(\alpha)_{X,Y}\colon \mathrm{Hom}_{\mathsf{Coll}(\mathfrak{p})}(X,Y) \longrightarrow \underbrace{\mathrm{Hom}_{\mathsf{Coll}(\mathfrak{q})}([\mathsf{Coll}(\alpha)](X),[\mathsf{Coll}(\alpha)](Y))}_{\stackrel{\text{def}}{=}\mathrm{Hom}_{\mathsf{Coll}(\mathfrak{q})}(X,Y)}$$

  of $\mathsf{Coll}(\alpha)$ at (X,Y) is defined as follows:

    - If $X, Y \in \mathrm{Obj}(\mathcal{C})$ or $X, Y \in \mathrm{Obj}(\mathcal{D})$, then we have

$$\mathsf{Coll}(\alpha)_{X,Y}(f) \stackrel{\text{def}}{=} f$$

    for each $f \in \mathrm{Hom}_{\mathsf{Coll}(\mathfrak{p})}(X,Y)$.

- If $X \in \mathrm{Obj}(\mathcal{C})$ and $Y \in \mathrm{Obj}(\mathcal{D})$, then

$$\mathrm{Coll}(\alpha)_{X,Y} \colon \underbrace{\mathrm{Hom}_{\mathrm{Coll}(\mathfrak{p})}(X,Y)}_{\stackrel{\text{def}}{=}\mathfrak{p}^X_Y} \longrightarrow \underbrace{\mathrm{Hom}_{\mathrm{Coll}(\mathfrak{q})}(X,Y)}_{\stackrel{\text{def}}{=}\mathfrak{q}^X_Y}$$

is defined by

$$\mathrm{Coll}(\alpha)_{X,Y}(f) \stackrel{\text{def}}{=} \alpha^X_Y;$$

- If $Y \in \mathrm{Obj}(\mathcal{C})$ and $X \in \mathrm{Obj}(\mathcal{D})$, then we have

$$\mathrm{Coll}(\alpha)_{X,Y}(f) \stackrel{\text{def}}{=} \mathrm{id}_{\emptyset}.$$

2. *Collages as Lax Colimits.* We have an isomorphism of categories

$$\mathrm{Coll}(\mathfrak{p}) \cong \mathrm{colim}^{\mathrm{lax}}(\mathfrak{p}),$$

functorial in $\mathfrak{p}$, where the above lax colimit is taken in the bicategory Prof.

3. *Profunctors vs. Collages.* We have an equivalence of categories

$$(\mathrm{Coll} \dashv \Gamma)\colon \quad \mathrm{Prof}(\mathcal{C},\mathcal{D}) \underset{\Gamma}{\overset{\mathrm{Coll}}{\rightleftarrows}} \mathrm{Cats}_{/\mathbb{I}}, \quad (\perp_{\mathrm{eq}})$$

where $\Gamma\colon \mathrm{Cats}_{/\mathbb{I}} \longrightarrow \mathrm{Prof}(\mathcal{C},\mathcal{D})$ is the functor sending a functor $\mathcal{E} \longrightarrow \mathbb{I}$ to the profunctor

$$\Gamma(\mathfrak{p})\colon \mathcal{C} \nrightarrow \mathcal{D}$$

given on objects by

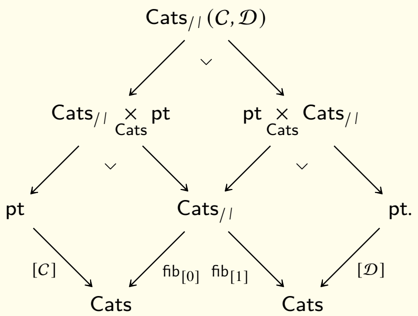
$$\Gamma(\mathfrak{p})^A_B \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathcal{E}}(A,B)$$

for each $A, B \in \mathrm{Obj}(\mathcal{E})$.

[1] Here $\mathrm{Cats}_{/\mathbb{I}}(\mathcal{C},\mathcal{D})$ is the category defined as the pullback

$$\mathrm{Cats}_{/\mathbb{I}}(\mathcal{C},\mathcal{D}) \stackrel{\text{def}}{=} \mathrm{pt} \underset{[\mathcal{C}],\mathrm{Cats},\mathrm{fib}_0}{\times} \mathrm{Cats}_{/\mathbb{I}} \underset{\mathrm{fib}_1,\mathrm{Cats},[\mathcal{D}]}{\times} \mathrm{pt},$$

as in the diagram



Proof 3.3.10 ▶ Proof of Proposition 3.3.9

Item 1: Functoriality

Omitted.

Item 2: Collages as Lax Colimits

See [Sch+17, Proposition 2.30].

Item 3: Profunctors vs. Collages

See [nLab23b, Proposition 2.4].

3.4 Properties of Prof

Proposition 3.4.1 ▶ Properties of the Bicategory of Profunctors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

1. *Self-Duality*. The bicategory Prof is self-dual: we have a biequivalence of bicategories

$$(-)^{\mathsf{op}} \colon \mathsf{Prof} \xrightarrow{\cong} \mathsf{Prof}^{\mathsf{op}}$$

where

- *Action on Objects*. The functor $(-)^{\mathsf{op}}$ sends categories to their opposites;
- *Action on* 1*-Morphisms*. The functor $(-)^{\mathsf{op}}$ sends profunctors to itself under the identification

$$\mathsf{Prof}(\mathcal{C}, \mathcal{D}) \stackrel{\text{def}}{=} \mathsf{Fun}(\mathcal{D}^{\mathsf{op}} \times \mathcal{C}, \mathsf{Sets}),$$

$$
\begin{aligned}
&\cong \mathsf{Fun}(\mathcal{C} \times \mathcal{D}^{\mathsf{op}}, \mathsf{Sets}),\\
&\stackrel{\text{def}}{=} \mathsf{Prof}(\mathcal{D}^{\mathsf{op}}, \mathcal{C}^{\mathsf{op}});
\end{aligned}
$$

- *Action on 2-Morphisms.* The functor $(-)^{\mathsf{op}}$ sends natural transformations between profunctors to themselves.

2. *Relation to* Cats. The co/representable profunctor constructions of Definitions 3.3.3 and 3.3.5 define embeddings of bicategories

$$
\begin{aligned}
\mathsf{Cats}^{\mathsf{op}} &\hookrightarrow \mathsf{Prof},\\
\mathsf{Cats}^{\mathsf{co}} &\hookrightarrow \mathsf{Prof}.
\end{aligned}
$$

3. *Equivalences in* Prof *and Cauchy Completions.* Every category is equivalent to its Cauchy completion in Prof.

4. *Equivalences in* Prof. The following conditions are equivalent:

   (a) The categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent in Prof.

   (b) The categories $\mathsf{PSh}(\mathcal{C})$ and $\mathsf{PSh}(\mathcal{D})$ are equivalent in Cats_2.

   (c) The Cauchy completions of $\mathcal{C}$ and $\mathcal{D}$ are equivalent in Cats_2.

5. *Adjunctions in* Prof. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. The following data are equivalent:

   (a) An adjunction in Prof from $\mathcal{C}$ to $\mathcal{D}$.

   (b) A functor from $\mathcal{C}$ to the Cauchy completion $\overline{\mathcal{D}}$ of $\mathcal{D}$.

   (c) A semifunctor from $\mathcal{C}$ to $\mathcal{D}$.

6. *As a Kleisli Bicategory.* We have a biequivalence of bicategories

$$\mathsf{Prof} \cong \mathsf{FreePsAlg}_{\mathsf{PSh}},$$

where PSh is the presheaf category relative pseudomonad of [Fio+18, Example 3.9].

7. *Closedness.* The bicategory Prof is a closed bicategory, where given a profunctor $\mathfrak{p} \colon \mathcal{C} \nrightarrow \mathcal{D}$ and a category $\mathcal{X}$:

   - *Right Kan Extensions.* The right adjoint

$$\mathsf{Ran}_{\mathfrak{p}} \colon \mathsf{Rel}(\mathcal{C}, \mathcal{X}) \longrightarrow \mathsf{Rel}(\mathcal{D}, \mathcal{X})$$

to the precomposition functor $\mathfrak{p}^* \colon \mathsf{Rel}(\mathcal{D}, \mathcal{X}) \longrightarrow \mathsf{Rel}(\mathcal{C}, \mathcal{X})$ is given by

$$\mathrm{Ran}_{\mathfrak{p}}(\mathfrak{q}) \stackrel{\text{def}}{=} \int_{A \in \mathcal{C}} \mathsf{Sets}\left(\mathfrak{p}_A^{-2}, \mathfrak{q}_A^{-1}\right)$$

for each $\mathfrak{q} \in \mathsf{Rel}(\mathcal{C}, \mathcal{X})$.

- *Right Kan Lifts.* The right adjoint to the postcomposition functor

$$\mathrm{Rift}_{\mathfrak{p}} \colon \mathsf{Rel}(\mathcal{X}, \mathcal{D}) \longrightarrow \mathsf{Rel}(\mathcal{X}, \mathcal{C})$$

to the postcomposition functor $\mathfrak{p}_* \colon \mathsf{Rel}(\mathcal{X}, \mathcal{C}) \longrightarrow \mathsf{Rel}(\mathcal{X}, \mathcal{D})$ is given by

$$\mathrm{Rift}_{\mathfrak{p}}(\mathfrak{q}) \stackrel{\text{def}}{=} \int_{B \in \mathcal{D}} \mathsf{Sets}\left(\mathfrak{p}_{-1}^{B}, \mathfrak{q}_{-2}^{B}\right)$$

for each $\mathfrak{q} \in \mathsf{Rel}(\mathcal{X}, \mathcal{D})$.

8. *Un/Straightening for Profunctors: Two-Sided Discrete Fibrations.* We have an equivalence of categories

$$\mathsf{Prof}(\mathcal{C}, \mathcal{D}) \cong \mathsf{DFib}(\mathcal{C}, \mathcal{D}).$$

Proof 3.4.2 ▶ Proof of Proposition 3.4.1

Item 1: Self-Duality

See [Lor21, Proposition 5.3.1].

Item 2: Relation to Cats

See [Lor21, Section 5.2].

Item 3: Equivalences in Prof and Cauchy Completions

See [Bor94a, Theorem 7.9.4].

Item 4: Equivalences in Prof

See [Bor94a, Theorem 7.9.4].

Item 5: Adjunctions in Prof

Omitted.

Item 6: As a Kleisli Bicategory

See [Fio+18, Example 4.2].

Item 7: Closedness

Omitted.

Item 8: Un/Straightening for Profunctors: Two-Sided Discrete Fibrations

See [Rie10, Theorem 2.3.2]

4 Monomorphisms

4.1 Foundations

Let $\mathcal{C}$ be a category.

Definition 4.1.1 ▶ Monomorphisms

A morphism $m\colon A \longrightarrow B$ of $\mathcal{C}$ is a **monomorphism** if for every commutative[1] diagram of the form

$$C \underset{g}{\overset{f}{\rightrightarrows}} A \xrightarrow{m} B,$$

we have $f = g$.

[1] That is, with $m \circ f = m \circ g$.

Example 4.1.2 ▶ Monomorphisms in Sets

Let $f\colon A \longrightarrow B$ be a function. The following conditions are equivalent:

1. The function f is injective.
2. The function f is a monomorphism in Sets.

Proof 4.1.3 ▶ Proof of Example 4.1.2

Suppose that f is a monomorphism and consider the following diagram:

$$\{*\} \underset{[y]}{\overset{[x]}{\rightrightarrows}} A \xrightarrow{f} B,$$

where $[x]$ and $[y]$ are the morphisms picking the elements x and y of A. Then $f(x) = f(y)$ iff $f \circ [x] = f \circ [y]$, implying $[x] = [y]$, and hence $x = y$. Therefore

f is injective.

Conversely, suppose that f is injective. Proceeding by contrapositive, we claim that given a pair of maps $g, h \colon C \rightrightarrows A$ such that $g \neq h$, then $f \circ g \neq f \circ h$. Indeed, as g and h are different maps, there exists must exist at least one element $x \in C$ such that $g(x) \neq h(x)$. But then we have $f(g(x)) \neq f(h(x))$, as f is injective. Thus $f \circ g \neq f \circ h$, and we are done.

Proposition 4.1.4 ▶ Properties of Monomorphisms

Let C be a category with pullbacks and $f \colon A \longrightarrow B$ be a morphism of C.

1. *Characterisations.* The following conditions are equivalent:
   (a) The morphism f is a monomorphism.
   (b) For each $X \in \mathrm{Obj}(C)$, the map of sets
   $$f_* \colon \mathrm{Hom}_{\mathsf{Sets}}(X, A) \longrightarrow \mathrm{Hom}_{\mathsf{Sets}}(X, B)$$
   is injective.
   (c) The kernel pair of f is trivial, i.e. we have
   $$A \times_B A \cong A, \qquad \begin{array}{ccc} A & \xrightarrow{\mathrm{id}_A} & A \\ {\scriptstyle \mathrm{id}_A}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ A & \xrightarrow[f]{} & B. \end{array}$$

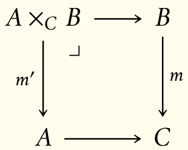
2. *Monomorphisms vs. Injective Maps.* Let
   - C be a concrete category;
   - 忘: $C \longrightarrow$ Sets be the forgetful functor from C to Sets;
   - $f \colon A \longrightarrow B$ be a morphism of C.

   If 忘 preserves pullbacks, then the following conditions are equivalent:
   (a) The morphism f is a monomorphism.
   (b) The morphism f is injective.

3. *Stability Properties.* The class of all monomorphisms of C is stable under the following operations:

(a) *Composition.* If f and g are monomorphisms, then so is $g \circ f$.[1]

(b) *Pullbacks.* Let



be a diagram in $\mathcal{C}$. If m is a monomorphism in $\mathcal{C}$, then so is m'.

4. *Morphisms From the Terminal Object Are Monomorphisms.* If $\mathcal{C}$ has a terminal object $\mathbb{1}_{\mathcal{C}}$, then every morphism of $\mathcal{C}$ from $\mathbb{1}_{\mathcal{C}}$ is a monomorphism.

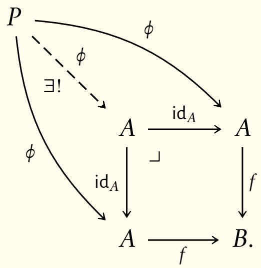
[1]Conversely, if $g \circ f$ is a monomorphism, then so is f.

PROOF 4.1.5 ▶ PROOF OF PROPOSITION 4.1.4

Item 1: Characterisations

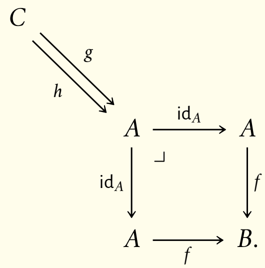
The equivalence between Items (a) and (b) is clear. We claim that Items (a) and (c) are equivalent:

1. *Item (a)* $\Longrightarrow$ *Item (c)*: Suppose that f is a monomorphism. Then A satisfies the universal property of the pullback:

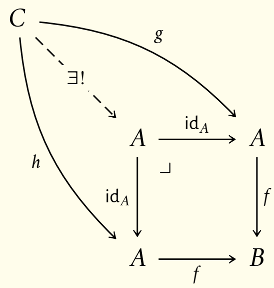


2. *Item (c)* $\Longrightarrow$ *Item (a)*: Suppose that $A \cong A \times_B A$ and let $g, h \colon C \rightrightarrows A$ be

a pair of morphisms. Consider the diagram



The universal property of the pullback says that there exists a unique morphism $C \longrightarrow A$ making the diagram



commute, which implies $g = h$. Therefore, f is a monomorphism.

Item 2: Monomorphisms vs. Injective Maps

Assume that f is injective. As the forgetful functor from $\mathcal{C}$ to Sets is faithful, we see that Proposition 4.2.2 together with **??** imply that f is a monomorphism.

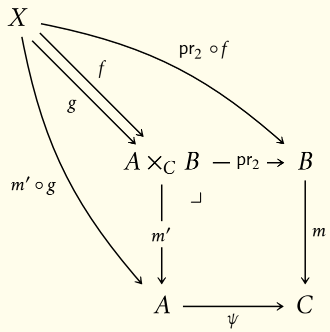
Conversely, assume that f is a monomorphism. As F preserves pullbacks, it also preserves kernel pairs. By **??**, we see that F preserves monomorphisms. Thus F_f is a monomorphism, and hence is injective by **??**.

Item 3: Stability Properties

Let $f, g \colon X \rightrightarrows A \times_C B$ be two morphisms such that the diagram

$$X \overset{f}{\underset{g}{\rightrightarrows}} A \times_C B \xrightarrow{m'} A$$

commutes. It follows that the diagram



also commutes. From the universal property of the pullback, it follows that there must be precisely one morphism from X to $A \times_C B$ making the above diagram commute. Thus $f = g$ and m' is a monomorphism.

Item 4: Morphisms From the Terminal Object Are Monomorphisms

Clear.

4.2 Monomorphism-Reflecting Functors

Definition 4.2.1 ▶ Monomorphism-Reflecting Functors

A functor $F \colon \mathcal{C} \longrightarrow \mathcal{D}$ **reflects monomorphisms** if, for each morphism f of $\mathcal{C}$, whenever F_f is a monomorphism, so is f.

Proposition 4.2.2 ▶ Faithful Functors Reflect Monomorphisms

Let $F \colon \mathcal{C} \longrightarrow \mathcal{D}$ be a functor. If F is faithful, then it reflects monomorphisms.

Proof 4.2.3 ▶ Proof of Proposition 4.2.2

Let $f \colon A \longrightarrow B$ be a morphism of $\mathcal{C}$ and suppose that $F_f \colon F_A \longrightarrow F_B$ is a monomorphism. Let $g, h \colon B \rightrightarrows C$ be two morphisms of $\mathcal{C}$ such that $g \circ f = h \circ f$. As F is faithful, we must have

$$F_g \circ F_f = F_{g \circ f} = F_{h \circ f} = F_h \circ F_f,$$

but as F_f is a monomorphism, it must be that $F_g = F_h$. Using the faithfulness of F again, we see that $g = h$. Therefore f is a monomorphism.

4.3 Split Monomorphisms

Let $\mathcal{C}$ be a category.

Definition 4.3.1 ▶ Split Monomorphisms

A morphism $f: A \longrightarrow B$ of $\mathcal{C}$ is a **split monomorphism**[1] if there exists a morphism $g: B \longrightarrow A$ of $\mathcal{B}$ such that[2]

$$g \circ f = \mathrm{id}_A.$$

[1] *Further Terminology:* Also called a **section**, or a **split monic** morphism.

[2] *Warning:* There exist monomorphisms which are not split monomorphisms, e.g. $\mathbb{Z}/2 \hookrightarrow \mathbb{Z}/4$ in Ring.

Proposition 4.3.2 ▶ Properties of Split Monomorphisms

Let $\mathcal{C}$ be a category.

1. *Split Monomorphisms are Monomorphisms.* If m is a split monomorphism, then m is a monomorphism.

Proof 4.3.3 ▶ Proof of Proposition 4.3.2

Item 1: Split Monomorphisms are Monomorphisms

Let $m: A \longrightarrow B$ be a split monomorphism of $\mathcal{C}$, let $e: B \longrightarrow A$ be a morphism of $\mathcal{C}$ with

$$e \circ m = \mathrm{id}_A,$$

and let $f, g: C \rightrightarrows A$ be two morphisms of $\mathcal{C}$ such that the diagram

$$C \underset{g}{\overset{f}{\rightrightarrows}} A \xrightarrow{m} B$$

commutes. Then we have

$$\begin{aligned} f &= \mathrm{id}_A \circ f \\ &= (e \circ m) \circ f \end{aligned}$$

$$
\begin{aligned}
&= e \circ (m \circ f) \\
&= e \circ (m \circ g) \\
&= (e \circ m) \circ g \\
&= \mathrm{id}_A \circ g \\
&= g,
\end{aligned}
$$

showing m to be a monomorphism.

5 Epimorphisms

5.1 Foundations

Let $\mathcal{C}$ be a category.

Definition 5.1.1 ▶ Epimorphisms

A morphism $f \colon A \longrightarrow B$ of $\mathcal{C}$ is an **epimorphism** if for every commutative[1] diagram of the form

$$A \xrightarrow{f} B \underset{h}{\overset{g}{\rightrightarrows}} C,$$

we have $g = h$.

[1] That is, with $g \circ f = h \circ f$.

Example 5.1.2 ▶ Epimorphisms in Sets

Let $f \colon A \longrightarrow B$ be a function. The following conditions are equivalent:

1. The function f is injective.
2. The function f is an epimorphism in Sets.

Proof 5.1.3 ▶ Proof of Example 5.1.2

Suppose that f is surjective and let $g, h \colon B \rightrightarrows C$ be morphisms such that $g \circ f = h \circ f$. Then for each $a \in A$, we have

$$g(f(a)) = h(f(a)),$$

but this implies that

$$g(b) = h(b)$$

for each $b \in B$, as f is surjective. Thus $g = h$ and f is an epimorphism.

To prove the converse, we proceed by contrapositive. So suppose that f is not surjective and consider the diagram

$$A \xrightarrow{f} B \underset{h}{\overset{g}{\rightrightarrows}} C,$$

where h is the map defined by $h(b) = 0$ for each $b \in B$ and g is the map defined by

$$g(b) = \begin{cases} 1 & \text{if } b \in \mathrm{Im}(f), \\ 0 & \text{otherwise.} \end{cases}$$

Then $h \circ f = g \circ f$, as $h(f(a)) = 1 = g(f(a))$ for each $a \in A$. However, for any $b \in B \setminus \mathrm{Im}(f)$, we have

$$g(b) = 0 \neq 1 = h(b).$$

Therefore $g \neq h$ and f is not an epimorphism.

Proposition 5.1.4 ▶ Properties of Epimorphisms

Let $\mathcal{C}$ be a category.

1. *Characterisations*. Let $\mathcal{C}$ be a category with pullbacks and $f\colon A \longrightarrow B$ be a morphism of $\mathcal{C}$. The following conditions are equivalent:

   (a) The morphism f is an epimorphism.

   (b) For each $X \in \mathrm{Obj}(\mathcal{C})$, the map of sets

   $$f^*\colon \mathrm{Hom}_{\mathsf{Sets}}(B, X) \longrightarrow \mathrm{Hom}_{\mathsf{Sets}}(A, X)$$

   is injective.

   (c) The cokernel pair of f is trivial, i.e. we have

   $$B \textstyle\coprod_A B \cong B \qquad \begin{array}{ccc} B & \longleftarrow & B \\ \uparrow & \ulcorner & \uparrow{\scriptstyle f} \\ B & \underset{f}{\longleftarrow} & A. \end{array}$$

2. *Epimorphisms vs. Surjective Maps.* Let

   - $\mathcal{C}$ be a concrete category;
   - $忘 \colon \mathcal{C} \longrightarrow \mathsf{Sets}$ be the forgetful functor from $\mathcal{C}$ to Sets;
   - $f \colon A \longrightarrow B$ be a morphism of $\mathcal{C}$.

   If 忘 preserves pushouts, then the following conditions are equivalent:

   (a) The morphism f is a epimorphism.

   (b) The morphism f is surjective.

3. *Stability Properties.* The class of all epimorphisms of $\mathcal{C}$ is stable under the following operations:

   (a) *Composition.* If f and g are epimorphisms, then so is $g \circ f$.[1]

   (b) *Pushouts.* Let

$$\begin{array}{ccc} A \coprod_C B & \longleftarrow & B \\ {\scriptstyle e'}\uparrow & \ulcorner & \uparrow{\scriptstyle e} \\ A & \longrightarrow & C \end{array}$$

   be a diagram in $\mathcal{C}$. If m is an epimorphism in $\mathcal{C}$, then so is e'.

4. *Morphisms to the Initial Object Are Monomorphisms.* If $\mathcal{C}$ has an initial object $\varnothing_{\mathcal{C}}$, then every morphism of $\mathcal{C}$ to $\varnothing_{\mathcal{C}}$ is a epimorphism.

[1] Conversely, if $g \circ f$ is a epimorphism, then so is g.

PROOF 5.1.5 ▶ PROOF OF PROPOSITION 5.1.4

This is dual to Proposition 4.1.4.

5.2 Regular Epimorphisms

PROPOSITION 5.2.1 ▶ PROPERTIES OF REGULAR EPIMORPHISMS

Let $\mathcal{C}$ be a category.

1. *Stability Under Pullbacks.* Consider the diagram

$$\begin{array}{ccc} A \times_C B & \longrightarrow & B \\ {\scriptstyle e'}\downarrow \lrcorner & & \downarrow{\scriptstyle e} \\ A & \longrightarrow & C \end{array}$$

in $\mathcal{C}$. If e is a regular epimorphism, then so is e'.

Proof 5.2.2 ▶ Proof of Proposition 5.2.1

Epimorphisms Need Not Be Stable Under Pullback.

Regular Epimorphisms Are Stable Under Pullback.

5.3 Effective Epimorphisms

Let $\mathcal{C}$ be a category.

Definition 5.3.1 ▶ Effective Epimorphisms

An epimorphism $f\colon A \longrightarrow B$ of $\mathcal{C}$ is **effective** if we have an isomorphism

$$B \cong \mathrm{CoEq}(A \times_B A \rightrightarrows A).$$

5.4 Split Epimorphisms

Let $\mathcal{C}$ be a category.

Definition 5.4.1 ▶ Retractions

A morphism $f\colon A \longrightarrow B$ of $\mathcal{C}$ is a **retraction**[1] if there is an arrow $g\colon B \longrightarrow A$ such that $f \circ g = \mathrm{id}_B$.

[1] *Further Terminology:* Also called a **split epimorphism**.

Proposition 5.4.2 ▶ Properties of Split Epimorphisms

Let $f\colon A \longrightarrow B$ be a morphism of $\mathcal{C}$.

1. Every split epimorphism is an epimorphism.[1]

[1] *Warning:* There are epimorphisms which are not split epimorphisms, however, e.g. $\mathbb{Z} \hookrightarrow \mathbb{Z}/2$.

Proof 5.4.3 ▶ Proof of Proposition 5.4.2

This is dual to ??.

6 Adjunctions

6.1 Foundations

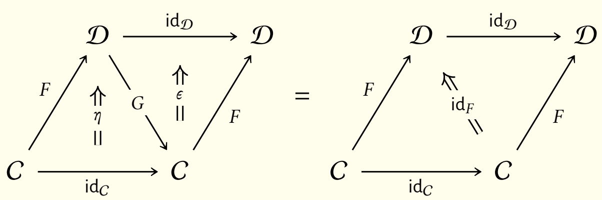
Let $\mathcal{C}$ and $\mathcal{D}$ be two categories.

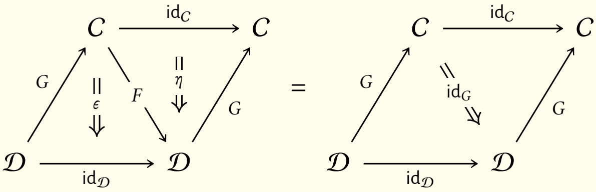
Definition 6.1.1 ▶ Adjunctions

An **adjunction**[1] is a quadruple (F, G, η, ϵ) consisting of

1. A functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$;
2. A functor $G\colon \mathcal{D} \longrightarrow \mathcal{C}$;
3. A natural transformation $\eta\colon \mathrm{id}_{\mathcal{C}} \Longrightarrow G \circ F$;
4. A natural transformation $\epsilon\colon F \circ G \Longrightarrow \mathrm{id}_{\mathcal{D}}$;

such that we have equalities

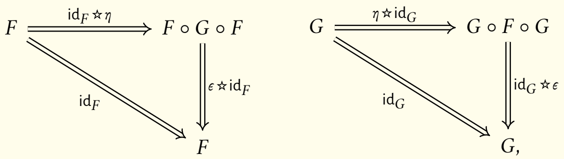




of pasting diagrams in Cats_2.[2]

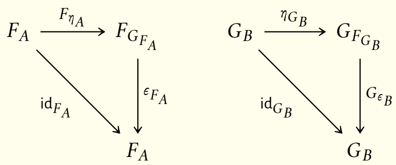
[1] *Further Terminology:* We also call (G, F) an **adjoint pair**, F a **left adjoint**, G a **right adjoint**, η the **unit** of the adjunction, and ϵ the **counit** of the adjunction.

[2] Equivalently, the diagrams



(6.1.1)

called the **left** and **right triangle identities**, commute, or, again equivalently, for each $A \in \mathrm{Obj}(\mathcal{C})$ and each $B \in \mathrm{Obj}(\mathcal{D})$, the diagrams

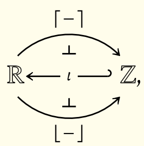


commute.

Example 6.1.2 ▶ Examples of Adjunctions

Here are some examples of adjunctions.

1. We have a triple adjunction

$(\lceil - \rceil \dashv \iota \dashv \lfloor - \rfloor)$: 

where $\mathbb{Z}$ and $\mathbb{R}$ are viewed as poset categories and $\iota \colon \mathbb{Z} \hookrightarrow \mathbb{R}$ is the canonical inclusion.

Proposition 6.1.3 ▶ Properties of Adjunctions

Let $F, L\colon \mathcal{C} \rightrightarrows \mathcal{D}$ and $G, R\colon \mathcal{D} \rightrightarrows \mathcal{C}$ be functors.

1. *Characterisations.* The following conditions are equivalent:

   (a) The pair (L, R) is an adjoint pair.

   (b) We have a natural isomorphism of (pro)functors[1]

   $$\mathsf{h}^L \cong \mathsf{h}_R.$$

   (c) For each $A \in \mathsf{Obj}(\mathcal{C})$ and each $B \in \mathsf{Obj}(\mathcal{D})$, we have an isomorphism

   $$\mathsf{Hom}_{\mathcal{D}}(L_A, B) \cong \mathsf{Hom}_{\mathcal{C}}(A, R_B)$$

   and the square below-left commutes iff the square below-right commutes:

   $$\begin{array}{ccc} L_A & \xrightarrow{f} & B \\ {\scriptstyle L_\phi}\downarrow & & \downarrow{\scriptstyle \psi} \\ L_{A'} & \xrightarrow[g]{} & B' \end{array} \qquad \Longleftrightarrow \qquad \begin{array}{ccc} A & \xrightarrow{f} & R_B \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle R_\psi} \\ A' & \xrightarrow[g]{} & R_{B'}. \end{array}$$

   (d) For each small category $\mathcal{K}$, we have an adjunction

   $$(L_* \dashv R_*)\colon \quad \mathsf{Fun}(\mathcal{K}, \mathcal{C}) \underset{R_*}{\overset{L_*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{Fun}(\mathcal{K}, \mathcal{D})$$

   as witnessed by a natural isomorphism

   $$\mathsf{Nat}(L \circ F, G) \cong \mathsf{Nat}(F, R \circ G)$$

   $$\left(\begin{array}{c} \mathcal{K} \xrightarrow{F} \mathcal{C} \xrightarrow{L} \mathcal{D} \\ \Downarrow \\ \mathcal{K} \xrightarrow{G} \mathcal{D} \end{array}\right) \quad \overset{\text{bij.}}{\longleftrightarrow} \quad \left(\begin{array}{c} \mathcal{K} \xrightarrow{F} \mathcal{C} \\ \Downarrow \\ \mathcal{K} \xrightarrow{G} \mathcal{D} \xrightarrow{R} \mathcal{C} \end{array}\right)$$

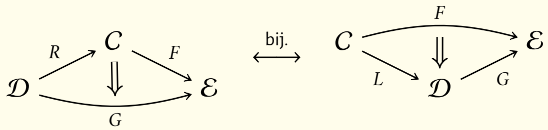
   natural in $\mathcal{K} \xrightarrow{F} \mathcal{C}$ and $\mathcal{K} \xrightarrow{G} \mathcal{D}$.

(e) For each locally small category $\mathcal{E}$, we have an adjunction

$$(R^* \dashv L^*)\colon\quad \mathsf{Fun}(\mathcal{C},\mathcal{E}) \underset{L^*}{\overset{R^*}{\rightleftarrows}}_{\perp} \mathsf{Fun}(\mathcal{D},\mathcal{E})$$

as witnessed by a natural isomorphism

$$\mathsf{Nat}(F \circ R, G) \cong \mathsf{Nat}(F, G \circ L)$$



natural in $\mathcal{C} \xrightarrow{F} \mathcal{E}$ and $\mathcal{D} \xrightarrow{G} \mathcal{E}$.

2. *Uniqueness.* If G admits left/right adjoints F_1 and F_2, then $F_1 \cong F_2$.[2]
3. *Stability Under Composition.* If $F_1 \dashv G_1$ and $F_2 \dashv G_2$, then $(F_2 \circ F_1) \dashv (G_2 \circ G_1)$:

$$\mathcal{C} \underset{G_1}{\overset{F_1}{\rightleftarrows}}_{\perp} \mathcal{D} \underset{G_2}{\overset{F_2}{\rightleftarrows}}_{\perp} \mathcal{E} \quad\rightsquigarrow\quad \mathcal{C} \underset{G_2 \circ G_1}{\overset{F_2 \circ F_1}{\rightleftarrows}}_{\perp} \mathcal{E}$$

4. *Interaction With Co/Limits.* The following statements are true:
   (a) **Left Adjoints Preserve Colimits (LAPC).** If F is a left adjoint, then F preserves all colimits that exist in $\mathcal{C}$.
   (b) **Right Adjoints Preserve Limits (RAPL).** If G is a right adjoint, then G preserves all limits that exist in $\mathcal{C}$.
5. *Interaction With Faithfulness.* Let (F, G, η, ϵ) be an adjunction. The following conditions are equivalent:
   (a) The functor F is faithful.
   (b) For each $A \in \mathsf{Obj}(\mathcal{C})$, the morphism

$$\eta_A \colon A \longrightarrow G_{F_A}$$

is a monomorphism.

Dually, the following conditions are equivalent:

(a) The functor G is faithful.

(b) For each $A \in \mathrm{Obj}(C)$, the morphism

$$\epsilon_A \colon F_{G_A} \longrightarrow A$$

is an epimorphism.

6. *Interaction With Fullness.* Let (F, G, η, ϵ) be an adjunction. The following conditions are equivalent:

(a) The functor F is full.

(b) For each $A \in \mathrm{Obj}(C)$, the morphism

$$\eta_A \colon A \longrightarrow G_{F_A}$$

is a split epimorphism.

Dually, the following conditions are equivalent:

(a) The functor G is full.

(b) For each $A \in \mathrm{Obj}(C)$, the morphism

$$\epsilon_A \colon F_{G_A} \longrightarrow A$$

is a split monomorphism.

7. *Interaction With Fully Faithfulness I.* Let (F, G, η, ϵ) be an adjunction. The following conditions are equivalent:

(a) The functor F is fully faithful.

(b) For each $A \in \mathrm{Obj}(C)$, the morphism

$$\eta_A \colon A \longrightarrow G_{F_A}$$

is an isomorphism.

(c) The following conditions are satisfied:

(i) The natural transformation

$$\mathrm{id}_F \star \eta \star \mathrm{id}_G \colon F \circ G \Longrightarrow F \circ G \circ F \circ G$$

is a natural isomorphism.

(ii) The functor F is conservative.

(iii) The functor G is essentially surjective.

Dually, the following conditions are equivalent:

(a) The functor G is fully faithful.

(b) For each $A \in \mathrm{Obj}(\mathcal{C})$, the morphism

$$\epsilon_A \colon F_{G_A} \longrightarrow A$$

is an isomorphism.

(c) The following conditions are satisfied:

(i) The natural transformation

$$\mathrm{id}_G \star \eta \star \mathrm{id}_F \colon G \circ F \Longrightarrow G \circ F \circ G \circ F$$

is a natural isomorphism.

(ii) The functor G is conservative.

(iii) The functor F is essentially surjective.

8. *Interaction With Fully Faithfulness II.* Let (F, G, η, ϵ) be an adjunction.

(a) If $G \circ F$ is fully faithful, then so is F.

(b) If $F \circ G$ is fully faithful, then so is G.

---

[1]That is, the following conditions are satisfied:

(i) *Bijection.* For each $A \in \mathrm{Obj}(\mathcal{C})$ and each $B \in \mathrm{Obj}(\mathcal{D})$, we have a bijection

$$\mathrm{Hom}_{\mathcal{D}}(L_A, B) \cong \mathrm{Hom}_{\mathcal{C}}(A, R_B).$$

(ii) *Naturality in $\mathcal{D}$.* For each morphism $g \colon B \longrightarrow B'$ of $\mathcal{D}$, the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(L_A, B) & \overset{\sim}{\dashrightarrow} & \mathrm{Hom}_{\mathcal{C}}(A, R_B) \\ {\scriptstyle h_g^{\mathrm{id}_{L_A}}} \downarrow & & \downarrow {\scriptstyle h_{R_g}^{\mathrm{id}_A}} \\ \mathrm{Hom}_{\mathcal{D}}(L_A, B') & \overset{\sim}{\dashrightarrow} & \mathrm{Hom}_{\mathcal{C}}(A, R_{B'}) \end{array}$$

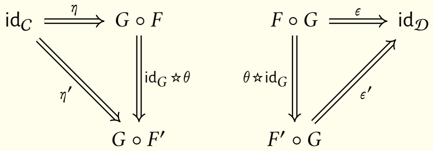
commutes.

(iii) *Naturality in $\mathcal{C}$.* For each morphism $f \colon A \longrightarrow A'$ of $\mathcal{C}$, the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(L_A, B) & \overset{\sim}{\dashrightarrow} & \mathrm{Hom}_{\mathcal{C}}(A, R_B) \\ {\scriptstyle h_{\mathrm{id}_B}^{L_f}} \downarrow & & \downarrow {\scriptstyle h_{\mathrm{id}_{R_B}}^{f}} \\ \mathrm{Hom}_{\mathcal{D}}(L_{A'}, B) & \overset{\sim}{\dashrightarrow} & \mathrm{Hom}_{\mathcal{C}}(A', R_B) \end{array}$$

commutes.

[2]Moreover, writing $\theta\colon F_1 \overset{\cong}{\Longrightarrow} F_2$ for this isomorphism, the diagrams



commute; see [Rie17, Proposition 4.4.1].

Proof 6.1.4 ▶ Proof of Proposition 6.1.3

Item 1: Adjunctions Via Hom-Functors

See [Rie17, Lemma 4.1.3 and Proposition 4.2.6].

Item 2: Uniqueness of Adjoints

This follows from the Yoneda lemma (Theorem 7.2.4) and its dual (Theorem 8.2.4).

Item 3: Stability Under Composition

See [Rie17, Proposition 4.4.4].

Item 4: Interaction With Limits and Colimits, Item (a)

[1]We prove Item (a) only, as Item (b) follows by duality (Limits and Colimits, Item 4 of Proposition 1.6.4). Indeed, let $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a functor admitting a right adjoint $G\colon \mathcal{D} \longrightarrow \mathcal{C}$. For each $Y \in \mathrm{Obj}(\mathcal{D})$, we have isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}}\left(F_{\mathrm{colim}(D)}, Y\right) &\cong \mathrm{Hom}_{\mathcal{D}}(\mathrm{colim}(D), G_Y) \\ &\cong \lim(\mathrm{Hom}_{\mathcal{D}}(D, G_Y)) \\ &\qquad \text{(Limits and Colimits, Item 11 of Proposition 1.6.4)} \\ &\cong \lim(\mathrm{Hom}_{\mathcal{D}}(F_D, Y)) \\ &\cong \mathrm{Hom}_{\mathcal{D}}(\mathrm{colim}(F_D), Y), \\ &\qquad \text{(Limits and Colimits, Item 11 of Proposition 1.6.4)} \end{aligned}$$

natural in $Y \in \mathrm{Obj}(\mathcal{D})$. The result then follows from Categories, **??**.

Item 4: Interaction With Limits and Colimits, Item (b)

This is dual to Item (a).

Item 5: Interaction With Faithfulness

See [Rie17, Lemma 4.5.13].

Item 6: Interaction With Fullness

See [Rie17, Lemma 4.5.13].

Item 7: Interaction With Fully Faithfulness I

See [Rie17, Lemma 4.5.13] and [Lor21, Proposition A.5.9].

Item 8: Interaction With Fully Faithfulness II

See [deJ20, Tag 0FWV], [Lor21, Proposition A.5.9], or [Low15, Propositions A.I.2 and A.I.3].

[1] *Reference*: See [Rie17, Theorem 4.5.2].

6.2 Existence Criteria for Adjoint Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

THEOREM 6.2.1 ▶ EXISTENCE CRITERIA FOR ADJOINT FUNCTORS

Let $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ and $G\colon \mathcal{D} \longrightarrow \mathcal{C}$ be functors.

1. *Via Comma Categories*. The following conditions are equivalent:
   (a) The functor F has a right adjoint.
   (b) For each $s \in \mathrm{Obj}(\mathcal{D})$, the comma category $F \downarrow s \cong \int^{\mathcal{C}} h^{F_-}_s$ has a terminal object.

   Dually, the following conditions are equivalent:

   (a) The functor G has a left adjoint F.
   (b) For each $s \in \mathrm{Obj}(\mathcal{C})$, the comma category $s \downarrow G \cong \int_{\mathcal{C}} h^s_{G_-}$ has an initial object.

   Moreover, when these conditions are satisfied, we have isomorphisms

$$\begin{aligned} F_A &\cong \lim_{A \to G_x} (x), \\ G_B &\cong \operatorname{colim}_{F_x \to G_B} (x), \end{aligned}$$

   natural in $A \in \mathrm{Obj}(\mathcal{C})$ and $B \in \mathrm{Obj}(\mathcal{D})$.

2. *The General Adjoint Functor Theorem*[1]. Suppose that
   (a) The category $\mathcal{D}$ has all limits and F commutes with them.
   (b) The category $\mathcal{C}$ is complete and locally small.
   (c) *The Solution Set Condition*. For each $X \in \mathrm{Obj}(\mathcal{D})$, there exist

(i) A small set I;

(ii) A set $\{A_i\}_{i\in I}$ of objects of $\mathcal{C}$;

(iii) A set $\{f_i\colon X \longrightarrow G_{A_i}\}$ of morphisms of $\mathcal{D}$;

such that, for each $i \in I$ and each morphism $f\colon X \longrightarrow G_A$, there exists a morphism $\phi_i\colon A_i \longrightarrow A$ of $\mathcal{C}$ together with a factorisation

$$X \xrightarrow{f_i} G_{A_i} \xrightarrow{G_{\phi_i}} G_A, \qquad f = G_{\phi_i} \circ f_i.$$

Then F has a left adjoint.

3. *The Special Adjoint Functor Theorem.* Suppose that

   (a) The category $\mathcal{D}$ has all limits and F commutes with them.

   (b) The category $\mathcal{C}$ is complete, locally small, and well-powered.

   (c) The category $\mathcal{C}$ has a small cogenerating set.

   Then F has a left adjoint.

4. *Freyd's Representability Theorem I.* Let $F\colon \mathcal{C} \longrightarrow \mathsf{Sets}$ be a functor. If[2]

   (a) The functor F commutes with limits;

   (b) The category $\mathcal{C}$ is complete and locally small;

   (c) *The Solution Set Condition.* There exists a set $\Phi \subset \mathrm{Obj}(\mathcal{C})$ such that, for each $c \in \mathrm{Obj}(\mathcal{C})$, there exist

   - $s \in \Phi$;
   - $y \in F_s$;
   - $f\colon s \longrightarrow c$ in $\mathrm{Hom}_{\mathsf{Sets}}(s, c)$;

   such that $F_{f(y)} = x$;

   then F is representable.

5. *Freyd's Representability Theorem II*[3]. Let $F\colon \mathcal{C} \longrightarrow \mathsf{Sets}$ be a functor. If

   (a) The functor F commutes with limits;

   (b) There exist

   - A collection $\{x_\alpha\}_{\alpha\in I}$ of object of $\mathcal{C}$;

- For each $\alpha \in I$, an element f_α of F_{x_α}

  such that for each $y \in \mathrm{Obj}(\mathcal{C})$ and each $g \in F_y$, there exists some $\alpha \in I$ and some morphism $\phi\colon x_i \longrightarrow y$ such that $F_\phi(f_\alpha) = g$;

then F is representable.

6. *Co/Totality.* Suppose that

   (a) The category $\mathcal{C}$ is locally small and cototal and $\mathcal{D}$ is locally small.

[1] *Further Terminology:* Also called **Freyd's adjoint functor theorem**.

[2] A nice application of this theorem is given in [MSE 276630], where it is used to abstractly show that Cats is cocomplete, avoiding the explicit construction of coequalisers in Cats given in **??**.

[3] This is the statement of Freyd's representability theorem as found in [deJ20, Tag 04HN].

PROOF 6.2.2 ▶ PROOF OF THEOREM 6.2.1

Item 1: Via Comma Categories

We claim that Items (a) and (b) are indeed equivalent:[1]

- *Item (a)* $\Longrightarrow$ *Item (b)*: Let F be a left adjoint of G. Then

$$\begin{aligned} s \downarrow G &\cong \int_{\mathcal{C}} h^s_{G_-} \\ &\cong \int_{\mathcal{C}} h^{F_s}_-, \end{aligned}$$

  where $h^s_{G_-}$ is corepresentable by F_s. By Fibred Categories, Item 10 of Proposition 9.4.1, it follows that the component $\eta_s\colon s \longrightarrow G_{F_s}$ of the unit of the adjunction $F \dashv G$ at s is an initial object of $s \downarrow G$.

- *Item (b)* $\Longrightarrow$ *Item (a)*: For each $s \in \mathrm{Obj}(\mathcal{D})$, write $\eta_s\colon s \longrightarrow G_{F_s}$ for an initial object of $s \downarrow G$. This gives us a map of sets

$$\begin{aligned} F\colon \mathrm{Obj}(\mathcal{C}) &\longrightarrow \mathrm{Obj}(\mathcal{D}) \\ s &\longmapsto F_s. \end{aligned}$$

  We now extend this map to a functor: given a morphism $f\colon s \longrightarrow s'$ of $\mathcal{C}$, we define $F_f\colon F_s \longrightarrow F_{s'}$ to be the unique morphism making the diagram

$$\begin{array}{ccc} s & \xrightarrow{\quad f \quad} & s' \\ {\scriptstyle \eta_s}\downarrow & & \downarrow{\scriptstyle \eta_{s'}} \\ G_{F_s} & \underset{G_{F_f}}{\dashrightarrow} & G_{F_{s'}} \end{array}$$

commute (which exists by the initiality of η_s). By the uniqueness of these morphisms, it follows that the assignment $s \mapsto F_s$ is indeed functorial. Moreover, we also obtain a natural transformation $\eta\colon \mathrm{id}_{\mathcal{C}} \Longrightarrow G \circ F$. We now define a natural transformation

$$\phi\colon \mathrm{Hom}_{\mathcal{D}}(F_{-}, b) \Longrightarrow \mathrm{Hom}_{\mathcal{C}}(-, G_b)$$

consisting of the collection

$$\left\{\phi_{s,b}\colon \mathrm{Hom}_{\mathcal{D}}(F_s, b) \Longrightarrow \mathrm{Hom}_{\mathcal{C}}(s, G_b)\right\}_{s\in\mathrm{Obj}(\mathcal{C})},$$

where $\phi_{s,b}$ is the map sending a morphism $g\colon F_s \longrightarrow b$ to the composition

$$s \xrightarrow{\eta_s} G_{F_s} \xrightarrow{G_g} G_b.$$

By the existence and uniqueness of morphisms from η_s to any other object $s \longrightarrow G_b$ in $s \downarrow G$, it follows that the maps $\phi_{s,b}$ are bijective, showing F to be a left adjoint of G.

Item 2: The General Adjoint Functor Theorem

See [Rie17, Theorem 4.6.3].

Item 3: The Special Adjoint Functor Theorem

See [Rie17, Theorem 4.6.10].

Item 4: Freyd's Representability Theorem I

See [Rie17, Theorem 4.6.15].

Item 5: Freyd's Representability Theorem II

See [deJ20, Tag `04HN`].

Item 6: Co/Totality

Omitted.

[1] *Reference:* [Rie17, Lemma 4.6.1].

6.3 Adjoint Strings

To avoid clutter, in this section we will abbreviate long compositions of functors. For instance, we write $f_1 \circ f_2 \circ f_3 \circ f_4$ as $f_1 f_2 f_3 f_4$. Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 6.3.1 ▶ Adjoint Strings

An **adjoint string of length** n[1] is an n-tuple $(f_1, \ldots, f_n)$ of functors between $\mathcal{C}$ and $\mathcal{D}$ such that

$$f_n \dashv f_{n+1}$$

for each $n \in \{1, \ldots, n-1\}$.

[1] *Further Terminology:* Also called an **adjoint** n**-tuple**.

Proposition 6.3.2 ▶ Properties of Adjoint Triples

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

1. *Adjoint Triples as Adjunctions Between Adjunctions.* An adjoint triple is equivalently an adjunction $(F \dashv G) \dashv (G \dashv H)$ between adjunctions. FIXME [nLab23a].[1]

2. *Adjunctions Induced by an Adjoint Triple.* A triple adjunction (f_1, f_2, f_3) gives rise to two more adjunctions

$$(f_2 f_1 \dashv f_2 f_3)\colon\ \mathcal{C} \underset{f_2 f_3}{\overset{f_2 f_1}{\underset{\bot}{\rightleftarrows}}} \mathcal{C}$$

and

$$(f_1 f_2 \dashv f_3 f_2)\colon\ \mathcal{D} \underset{f_3 f_2}{\overset{f_1 f_2}{\underset{\bot}{\rightleftarrows}}} \mathcal{D}$$

where $f_2 f_1$ and $f_2 f_3$ are monads in $\mathcal{C}$ and $f_1 f_2$ and $f_3 f_2$ are comonads in $\mathcal{D}$.

[1] [nLab23a] suggests writing

$$\begin{array}{ccc} f_1 & \dashv & f_2 \\ \bot & & \bot \\ f_2 & \dashv & f_3 \end{array}$$

to denote the adjunctions $(f_1 \dashv f_2 \dashv f_3)$ and $(f_1 f_2) \dashv (f_2 f_3)$ simultaneously; the first horizontally and the latter vertically.

Proof 6.3.3 ▶ Proof of Proposition 6.3.2

Item 1: Adjoint Triples as Adjunctions Between Adjunctions

Omitted.

Item 2: Adjunctions Induced by an Adjoint Triple

Omitted.

PROPOSITION 6.3.4 ▶ PROPERTIES OF ADJOINT QUADRUPLES

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

1. *Adjunctions Induced by a Quadruple Adjunction.* An adjoint quadruple $(f_1 \dashv f_2 \dashv f_3 \dashv f_4)$ gives rise to two adjoint triples

$$(f_2f_1 \dashv f_2f_3 \dashv f_4f_3)\colon \quad \mathcal{C} \ \substack{\xrightarrow{\ f_2f_1\ } \\ \perp \\ \xleftarrow{\ f_2f_3\ } \\ \perp \\ \xrightarrow{\ f_4f_3\ }} \ \mathcal{C}$$

and

$$(f_1f_2 \dashv f_3f_2 \dashv f_3f_4)\colon \quad \mathcal{D} \ \substack{\xrightarrow{\ f_1f_2\ } \\ \perp \\ \xleftarrow{\ f_3f_2\ } \\ \perp \\ \xrightarrow{\ f_3f_4\ }} \ \mathcal{D}$$

and six adjunctions

$$(f_1f_2f_3 \dashv f_4f_3f_2)\colon \quad \mathcal{C} \ \underset{f_4f_3f_2}{\overset{f_1f_2f_3}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \ \mathcal{D} \qquad (f_3f_2f_1 \dashv f_2f_3f_4)\colon \quad \mathcal{C} \ \underset{f_2f_3f_4}{\overset{f_3f_2f_1}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \ \mathcal{D}$$

$$(f_2f_3f_2f_1 \dashv f_2f_3f_4f_3)\colon \quad \mathcal{C} \ \underset{f_2f_3f_4f_3}{\overset{f_2f_3f_2f_1}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \ \mathcal{C} \qquad (f_3f_2f_1f_2 \dashv f_3f_2f_3f_4)\colon \quad \mathcal{C} \ \underset{f_3f_2f_3f_4}{\overset{f_3f_2f_1f_2}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \ \mathcal{C}$$

$$(f_2f_1f_2f_3 \dashv f_4f_3f_2f_3)\colon \quad \mathcal{D} \underset{f_4f_3f_2f_3}{\overset{f_2f_1f_2f_3}{\rightleftarrows}} \mathcal{D} \qquad (f_1f_2f_3f_2 \dashv f_3f_4f_3f_2)\colon$$

$$\mathcal{D} \underset{f_3f_4f_3f_2}{\overset{f_1f_2f_3f_2}{\rightleftarrows}} \mathcal{D}$$

where f_2f_1, f_2f_3, f_4f_3, $f_2f_3f_2f_1$, $f_2f_3f_4f_3$, $f_3f_2f_1f_2$, and $f_3f_2f_3f_4$ are monads in $\mathcal{C}$ and f_1f_2, f_3f_2, f_3f_4, $f_2f_1f_2f_3$, $f_4f_3f_2f_3$, $f_1f_2f_3f_2$, and $f_3f_4f_3f_2$ are comonads in $\mathcal{D}$.

Proof 6.3.5 ▶ Proof of Proposition 6.3.4

Item 1: Adjunctions Induced by a Quadruple Adjunction

Omitted. ∎

Proposition 6.3.6 ▶ Adjunctions Induced by an Adjoint String of Length n

Let $(f_1 \dashv \cdots \dashv f_n)\colon \mathcal{C} \rightleftarrows \mathcal{D}$ be an adjoint string.

1. For each $k \in \mathbb{N}$ with $1 \leq k \leq n-2$, we have 2 induced adjoint strings

   $f_1f_2 \cdots f_{n-k}f_{n-k+1} \dashv f_{n-k+2}f_{n-k+1} \cdots f_3f_2 \dashv \cdots \dashv f_{k-1}f_k \cdots f_{n-2}f_{n-1} \dashv f_nf_{n-1} \cdots f_{k+1}f_k$

   $f_{n-k+1}f_{n-k} \cdots f_2f_1 \dashv f_2f_3 \cdots f_{n-k+1}f_{n-k+2} \dashv \cdots \dashv f_{n-1}f_{n-2} \cdots f_kf_{k-1} \dashv f_kf_{k+1} \cdots f_{n-1}f_n$

   of length $n-k$.

2. Inductively applying Item 1 to the induced adjoint strings, we get (including the 2 adjoint strings of Item 1) $2 \cdot 3^{n-k-1}$ adjoint strings of length k[1], for a grand total of

   $$\sum_{k=2}^{n-1} 2(k-1) \cdot 3^{n-k-1} = \frac{1}{6}(3^n + 3) - n$$

   adjunctions.[2]

3. In particular:

   (a) An adjoint triple induces 2 adjoint pairs.

   (b) An adjoint quadruple induces

- 2 adjoint triples,
- 6 adjoint pairs,

for a grand total of 10 adjunctions.

(c) An adjoint quintuple induces

- 2 adjoint quadruples,
- 6 adjoint triples,
- 18 adjoint pairs,

for a grand total of 36 adjunctions.

(d) An adjoint sextuple induces

- 2 adjoint quintuples,
- 6 adjoint quadruples,
- 18 adjoint triples,
- 54 adjoint pairs,

for a grand total of 116 adjunctions.

(e) An adjoint septuple induces

- 2 adjoint sextuples,
- 6 adjoint quintuples,
- 18 adjoint quadruples,
- 54 adjoint triples,
- 162 adjoint pairs,

for a grand total of 358 adjunctions.

---

[1] These need not be unique.

[2] E.g. we have 4 adjoint strings of length $n-2$, such as

$$f_2f_3f_2f_1 \dashv f_2f_3f_4f_3 \dashv \cdots \dashv f_kf_{k+1}f_kf_{k-1} \dashv f_kf_{k+1}f_{k+2}f_{k+1} \dashv \cdots \dashv f_{n-2}f_{n-1}f_{n-2}f_{n-1} \dashv f_{n-2}f_{n-1}f_nf_{n-1}.$$

Proof 6.3.7 ▶ Proof of Proposition 6.3.6

Omitted. ∎

6.4 Reflective Subcategories

Let C be a category.

Definition 6.4.1 ▶ Reflective Subcategories

A subcategory $\mathcal{C}_0$ of $\mathcal{C}$ is **reflective** if the inclusion functor $i\colon \mathcal{C}_0 \hookrightarrow \mathcal{C}$ of $\mathcal{C}_0$ into $\mathcal{C}$ admits a left adjoint $L\colon \mathcal{C} \longrightarrow \mathcal{C}_0$.[1]

[1] *Further Terminology*: The functor L is called the **reflector** or **localisation** of the adjunction $L \dashv i$.

Example 6.4.2 ▶ Examples of Reflective Subcategories

Here are some examples of reflective subcategories

1. $\mathsf{CHaus} \hookrightarrow \mathsf{Top}$ *([Rie17, Example 4.5.14, (i)])*. The category CHaus is a reflective subcategory of Top, as witnessed by the adjunction

$$(\beta \dashv \iota)\colon \quad \mathsf{Top} \underset{\iota}{\overset{\beta}{\rightleftarrows}} \mathsf{CHaus},$$

   of Topological Spaces, **??** of **??**.

2. $\mathsf{CMon} \hookrightarrow \mathsf{Mon}$. The category CMon is a reflective subcategory of Ab, as witnessed by the adjunction

$$\left((-)^{\mathrm{ab}} \dashv \iota\right)\colon \quad \mathsf{Mon} \underset{\iota}{\overset{(-)^{\mathrm{ab}}}{\rightleftarrows}} \mathsf{CMon}$$

   of Monoids, **??** of **??**.

3. $\mathsf{Ab} \hookrightarrow \mathsf{Grp}$ *([Rie17, Example 4.5.14, (ii)])*. The category Ab is a reflective subcategory of Grp, as witnessed by the adjunction

$$\left((-)^{\mathrm{ab}} \dashv \iota\right)\colon \quad \mathsf{Grp} \underset{\iota}{\overset{(-)^{\mathrm{ab}}}{\rightleftarrows}} \mathsf{Ab}$$

   of Groups, **??** of **??**.

4. $\mathsf{Ab}^{\mathrm{tf}} \hookrightarrow \mathsf{Ab}$ *([Rie17, Example 4.5.14, (iii)])*. The full subcategory $\mathsf{Ab}^{\mathrm{tf}}$ of Ab spanned by the torsion-free abelian groups is reflective in Ab. This is witnessed by the adjunction

$$\left((-)^{\mathrm{tf}} \dashv \iota\right)\colon \quad \mathsf{Ab} \underset{\iota}{\overset{(-)^{\mathrm{tf}}}{\rightleftarrows}} \mathsf{Ab}^{\mathrm{tf}},$$

   where $(-)^{\mathrm{tf}}\colon \mathsf{Ab} \longrightarrow \mathsf{Ab}^{\mathrm{tf}}$ is the functor defined on objects by sending an abelian group A to the quotient $A/\mathrm{Tors}(A)$, where $\mathrm{Tors}(A)$ is the torsion subgroup of A.

5. $\mathsf{Mod}_S \hookrightarrow \mathsf{Mod}_R$ *([Rie17, Example 4.5.14, (iv)])*. Let $\phi\colon R \longrightarrow S$ be a morphism of rings. Then ϕ^* is full iff ϕ is an epimorphism, in which case the adjunction

$$(S \otimes_R (-) \dashv \phi^*)\colon \quad \mathsf{Mod}_S \underset{\phi^*}{\overset{S\otimes_R(-)}{\rightleftarrows}} \mathsf{Mod}_R$$

witnesses Mod_S as a reflective subcategory of Mod_R.

6. $\mathsf{Shv}(C) \hookrightarrow \mathsf{PSh}(C)$ *([Rie17, Example 4.5.14, (v)])*. The category $\mathsf{Shv}(C)$ of sheaves on a site C is a reflective subcategory of $\mathsf{PSh}(C)$, as witnessed by the adjunction

$$\Big((-)^\# \dashv \iota\Big)\colon \quad \mathsf{PSh}(C) \underset{\iota}{\overset{(-)^\#}{\rightleftarrows}} \mathsf{Shv}(C),$$

of Sites, Section 5.5.

7. $\mathsf{Cats} \hookrightarrow \mathsf{sSets}$ *([Rie17, Example 4.5.14, (v)])*. The category Cats is a reflective subcategory of sSets, as witnessed by the adjunction

$$(\mathsf{Ho} \dashv \mathsf{N}_\bullet)\colon \quad \mathsf{sSets} \underset{\mathsf{N}_\bullet}{\overset{\mathsf{Ho}}{\rightleftarrows}} \mathsf{Cats}$$

of Quasicategories, Item 3 of Proposition 1.5.4.

Proposition 6.4.3 ▶ Properties of Reflective Subcategories

Let C_0 be a reflective subcategory of C.

1. *Characterisations.* Let

$$(L \dashv \iota)\colon \quad C \underset{\iota}{\overset{L}{\rightleftarrows}} \mathcal{D}$$

be an adjunction. The following conditions are equivalent:

(a) The functor ι is fully faithful.

(b) The counit $\epsilon\colon L \circ \iota \Longrightarrow \mathrm{id}_{\mathcal{D}}$ is a natural isomorphism.

(c) The following conditions are satisfied:

(i) The monad $(\iota \circ L, \mathrm{id}_\iota \star \epsilon \star \mathrm{id}_L, \eta)$ associated to the adjunction $L \dashv \iota$ is idempotent.

(ii) The functor ι is conservative.

(iii) The functor L is essentially surjective.

(d) The functor L is the Gabriel–Zisman localisation of C with respect to the class S given by

$$S \stackrel{\text{def}}{=} \{f \in \mathrm{Mor}(C) \mid L(f) \text{ is an isomorphism in } \mathcal{D}\}.$$

(e) The functor L is dense.

2. *Interaction With Limits.* The inclusion $C_0 \hookrightarrow C$ creates all limits which exist in C.

3. *Interaction With Colimits.* The category C_0 admits all colimits that exist in C: given a diagram $D \colon \mathcal{I} \longrightarrow C_0$ in C_0, if $\mathrm{colim}(i \circ D)$ exists in C, then $\mathrm{colim}(D)$ exists in C_0 and we have

$$\mathrm{colim}(D) \cong L(\mathrm{colim}(i \circ D)).$$

Proof 6.4.4 ▶ Proof of Proposition 6.4.3

Item 1: Characterisations

See [GZ67, Proposition 1.3] and [Ulm68, Theorem 1.13].

Item 2: Interaction With Limits

See [Rie17, Proposition 4.5.15].

Item 3: Interaction With Colimits

See [Rie17, Proposition 4.5.15].

6.5 Coreflective Subcategories

Let C be a category.

Definition 6.5.1 ▶ Coreflective Subcategories

A subcategory $\mathcal{C}_0$ of $\mathcal{C}$ is **coreflective** if the inclusion functor $i\colon \mathcal{C}_0 \hookrightarrow \mathcal{C}$ of $\mathcal{C}_0$ into $\mathcal{C}$ admits a right adjoint $R\colon \mathcal{C} \longrightarrow \mathcal{C}_0$.[1]

[1] *Further Terminology:* The functor L is called the **coreflector** or **colocalisation** of the adjunction $i \dashv R$.

7 The Yoneda Lemma

7.1 Presheaves

Let $\mathcal{C}$ be a category.

Definition 7.1.1 ▶ Presheaves on a Category

A **presheaf on** $\mathcal{C}$ is a functor $\mathcal{F}\colon \mathcal{C}^{\mathsf{op}} \longrightarrow \mathsf{Sets}$.

Definition 7.1.2 ▶ The Category of Presheaves on a Category

The **category of presheaves on** $\mathcal{C}$ is the category $\mathsf{PSh}(\mathcal{C})$ defined by

$$\mathsf{PSh}(\mathcal{C}) \overset{\text{def}}{=} \mathsf{Fun}(\mathcal{C}^{\mathsf{op}}, \mathsf{Sets}).$$

Remark 7.1.3 ▶ Unwinding Definition 7.1.2

In detail, the **category of presheaves on** $\mathcal{C}$ is the category $\mathsf{PSh}(\mathcal{C})$ where

- *Objects.* The objects of $\mathsf{PSh}(\mathcal{C})$ are presheaves on $\mathcal{C}$;
- *Morphisms.* A morphism of $\mathsf{PSh}(\mathcal{C})$ from $\mathcal{F}$ to $\mathcal{G}$ is a natural transformation $\alpha\colon \mathcal{F} \Longrightarrow \mathcal{G}$;
- *Identities.* For each $\mathcal{F} \in \mathrm{Obj}(\mathsf{PSh}(\mathcal{C}))$, the unit map

  $$\mathbb{1}^{\mathsf{PSh}(\mathcal{C})}_{\mathcal{F}}\colon \mathrm{pt} \longrightarrow \mathrm{Nat}(\mathcal{F}, \mathcal{F})$$

  of $\mathsf{PSh}(\mathcal{C})$ at $\mathcal{F}$ is defined by

  $$\mathrm{id}^{\mathsf{PSh}(\mathcal{C})}_{\mathcal{F}} \overset{\text{def}}{=} \mathrm{id}_{\mathcal{F}};$$

- *Composition.* For each $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathrm{Obj}(\mathsf{PSh}(\mathcal{C}))$, the composition map

  $$\circ^{\mathsf{PSh}(\mathcal{C})}_{\mathcal{F},\mathcal{G},\mathcal{H}}\colon \mathrm{Nat}(\mathcal{G}, \mathcal{H}) \times \mathrm{Nat}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Nat}(\mathcal{F}, \mathcal{H})$$

of $\mathsf{PSh}(\mathcal{C})$ at $(\mathcal{F}, \mathcal{G}, \mathcal{H})$ is defined by

$$\beta \circ^{\mathsf{PSh}(\mathcal{C})}_{\mathcal{F},\mathcal{G},\mathcal{H}} \alpha \stackrel{\text{def}}{=} \beta \circ \alpha.$$

7.2 Representable Presheaves

Let $\mathcal{C}$ be a category, let $U, V \in \mathrm{Obj}(\mathcal{C})$, and let $f \colon U \longrightarrow V$ be a morphism of $\mathcal{C}$.

Definition 7.2.1 ▶ The Representable Presheaf Associated to an Object

The **representable presheaf associated to** U is the presheaf $h_U \colon \mathcal{C}^{\mathsf{op}} \longrightarrow \mathsf{Sets}$ on $\mathcal{C}$ where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathcal{C})$, we have

$$h_U(A) \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(A, U);$$

- *Action on Morphisms.* For each morphism $f \colon A \longrightarrow B$ of $\mathcal{C}$, the image

$$h_U(f) \colon \underbrace{h_U(B)}_{\stackrel{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(B,U)} \longrightarrow \underbrace{h_U(A)}_{\stackrel{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(A,U)}$$

of f by h_U is defined by

$$h_U(f) \stackrel{\text{def}}{=} f^*.$$

Definition 7.2.2 ▶ Representable Presheaves

A presheaf $\mathcal{F} \colon \mathcal{C}^{\mathsf{op}} \longrightarrow \mathsf{Sets}$ is **representable** if $\mathcal{F} \cong h_U$ for some $U \in \mathrm{Obj}(\mathcal{C})$.[1]

[1] In such a case, we call U a **representing object** for $\mathcal{F}$.

Definition 7.2.3 ▶ Representable Natural Transformations

The **representable natural transformation associated to** f is the natural transformation $h_f\colon h_U \Longrightarrow h_V$ consisting of the collection

$$\left\{ h_{f|A}\colon \underbrace{h_U(A)}_{\overset{\text{def}}{=}\mathrm{Hom}_{\mathcal{C}}(A,U)} \longrightarrow \underbrace{h_V(A)}_{\overset{\text{def}}{=}\mathrm{Hom}_{\mathcal{C}}(A,V)} \right\}_{A\in\mathrm{Obj}(\mathcal{C})}$$

where

$$h_{f|A} \overset{\text{def}}{=} f_*.$$

Theorem 7.2.4 ▶ The Yoneda Lemma

Let $\mathcal{F}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{Sets}$ be a presheaf on $\mathcal{C}$. We have a bijection

$$\mathrm{Nat}(h_A, \mathcal{F}) \cong \mathcal{F}_A,$$

natural in $A \in \mathrm{Obj}(\mathcal{C})$, determining a natural isomorphism of functors

$$\mathrm{Nat}\big(h_{(-)}, \mathcal{F}\big) \cong \mathcal{F}.$$

Proof 7.2.5 ▶ Proof of Theorem 7.2.4

The Natural Transformation $\mathrm{ev}_{(-)}\colon \mathrm{Nat}\big(h_{(-)}, \mathcal{F}\big) \Longrightarrow \mathcal{F}$

Let $\mathrm{ev}_{(-)}\colon \mathrm{Nat}\big(h_{(-)}, \mathcal{F}\big) \Longrightarrow \mathcal{F}$ be the natural transformation consisting of the collection

$$\{\mathrm{ev}_A\colon \mathrm{Nat}(h_A, \mathcal{F}) \longrightarrow \mathcal{F}(A)\}_{A\in\mathrm{Obj}(\mathcal{C})}$$

with

$$\mathrm{ev}_A(\alpha) = \alpha_A(\mathrm{id}_A)$$

for each $\alpha\colon h_A \Longrightarrow \mathcal{F}$ in $\mathrm{Nat}(h_A, \mathcal{F})$.

The Natural Transformation $\xi_{(-)}\colon \mathcal{F} \Longrightarrow \mathrm{Nat}\big(h_{(-)}, \mathcal{F}\big)$

Let $\xi_{(-)}\colon \mathcal{F} \Longrightarrow \mathrm{Nat}\big(h_{(-)}, \mathcal{F}\big)$ be the natural transformation consisting of the collection

$$\{\xi_A\colon \mathcal{F}(A) \longrightarrow \mathrm{Nat}(h_A, \mathcal{F})\}_{A\in\mathrm{Obj}(\mathcal{C})}$$

where $\xi_A \colon \mathcal{F}(A) \longrightarrow \mathrm{Nat}(h_A, \mathcal{F})$ is the map sending an element f of $\mathcal{F}(X)$ to the natural transformation

$$\xi_{A,f} \colon h_A \Longrightarrow \mathcal{F}$$

consisting of the collection

$$\left\{ \left(\xi_{A,f}\right)_U \colon h_A(U) \longrightarrow \mathcal{F}(U) \right\}_{A \in \mathrm{Obj}(C)}$$

where $\left(\xi_{A,f}\right)_U \colon h_A(U) \longrightarrow \mathcal{F}(U)$ is the morphism given by

$$\begin{aligned} \left(\xi_{A,f}\right)_U \colon h_A(U) &\longrightarrow \mathcal{F}(U) \\ (h \colon U \longrightarrow A) &\longmapsto \mathcal{F}(h)(f) \end{aligned}$$

for each $f \colon U \longrightarrow A$ in $h_A(U)$.

$\mathrm{ev}_{(-)} \circ \xi_{(-)} = \mathrm{id}_{\mathcal{F}}$

Let $f \in \mathcal{F}(X)$. We have

$$\begin{aligned} \left(\xi_{A,f}\right)_U(\mathrm{id}_U) &= \mathcal{F}(\mathrm{id}_U)(f), \\ &= \mathrm{id}_{\mathcal{F}(U)}(f) \\ &= f. \end{aligned}$$

$\xi_{(-)} \circ \mathrm{ev}_{(-)} = \mathrm{id}_{\mathrm{Nat}(h_{(-)}, \mathcal{F})}$

Let $\alpha \colon h_A \Longrightarrow \mathcal{F} \in \mathrm{Nat}(h_A, \mathcal{F})$ and consider the diagram

$$\begin{array}{ccc} \mathrm{Hom}_C(A, A) & \xrightarrow{h_f} & \mathrm{Hom}_C(A, X) \\ \Big\downarrow{\scriptstyle \xi_A} & & \Big\downarrow{\scriptstyle \xi_X} \\ \mathcal{F}(A) & \xrightarrow[\mathcal{F}(f)]{} & \mathcal{F}(X) \end{array}$$

defined on elements by

$$\begin{array}{ccc} \mathrm{id}_A & \longmapsto & f \\ \big\downarrow & & \big\downarrow \\ u & \longmapsto & \mathcal{F}(f)(u) = \xi_X(f). \end{array}$$

Then it is clear that the natural transformation ξ is determined by $\xi_A(\mathrm{id}_A) = u$, since we must have

$$\xi_X(f) = \mathcal{F}(f)(u)$$

for each $X \in \mathrm{Obj}(\mathcal{C})$ and each morphism $f\colon A \longrightarrow X$ of $\mathcal{C}$.

7.3 The Yoneda Embedding

Definition 7.3.1 ▶ The Covariant Yoneda Embedding

The **covariant Yoneda embedding of $\mathcal{C}$**[1] is the functor[2]

$$よ_{\mathcal{C}}\colon \mathcal{C} \hookrightarrow \mathsf{PSh}(\mathcal{C})$$

where

- *Action on Objects.* For each $U \in \mathrm{Obj}(\mathcal{C})$, we have

$$よ(U) \stackrel{\text{def}}{=} h_U;$$

- *Action on Morphisms.* For each morphism $f\colon U \longrightarrow V$ of $\mathcal{C}$, the image

$$よ(f)\colon よ(U) \longrightarrow よ(V)$$

  of f by よ is defined by

$$よ(f) \stackrel{\text{def}}{=} h_f.$$

[1] *Further Terminology:* Also called simply the **Yoneda embedding**.
[2] *Further Notation:* Also written $h_{(-)}$, or simply よ.

Proposition 7.3.2 ▶ Properties of the Yoneda Embedding

Let $\mathcal{C}$ be a category.

1. *Fully Faithfulness.* The Yoneda embedding is fully faithful.[1]

2. *Preservation and Reflection of Isomorphisms.* Let $A, B \in \mathrm{Obj}(\mathcal{C})$. The following conditions are equivalent:

   (a) We have $A \cong B$.

   (b) We have $h_A \cong h_B$.

(c) We have $h^A \cong h^B$.

3. *Uniqueness of Representing Objects Up to Isomorphism.* Let $\mathcal{F}\colon \mathcal{C}^{\mathsf{op}} \longrightarrow \mathsf{Sets}$ be a presheaf. If there exist objects A and B of $\mathcal{C}$ such that we have

$$\begin{aligned} h_A &\cong \mathcal{F}, \\ h_B &\cong \mathcal{F}, \end{aligned}$$

then $A \cong B$.

4. *As a Free Cocompletion: The Universal Property.* The pair $(\mathsf{PSh}(\mathcal{C}), \text{よ})$ consisting of

- The category $\mathsf{PSh}(\mathcal{C})$ of presheaves on $\mathcal{C}$;
- The Yoneda embedding $\text{よ}\colon \mathcal{C} \hookrightarrow \mathsf{PSh}(\mathcal{C})$ of $\mathcal{C}$ into $\mathsf{PSh}(\mathcal{C})$;

satisfies the following universal property:

(**UP**) Given another pair $(\mathcal{A}, F)$ consisting of

- A cocomplete category $\mathcal{A}$;
- A cocontinuous functor $F\colon \mathcal{C} \longrightarrow \mathcal{A}$;

there exists a cocontinuous functor $\mathsf{PSh}(\mathcal{C}) \overset{\exists!}{\longrightarrow} \mathcal{A}$, unique up to natural isomorphism, making the diagram

$$\begin{array}{ccc} & & \mathsf{PSh}(\mathcal{C}) \\ & \overset{\text{よ}}{\nearrow} \;\; \overset{\sim}{\Nearrow} & \Big\downarrow {\scriptstyle \exists!} \\ \mathcal{C} & \xrightarrow[F]{} & \mathcal{A} \end{array}$$

commute, again up to natural isomorphism.

5. *As a Free Cocompletion: 2-Adjointness.* We have a 2-adjunction

$$(\mathsf{PSh} \dashv \iota)\colon \quad \mathsf{Cats} \underset{\iota}{\overset{\mathsf{PSh}}{\rightleftarrows}} \mathsf{Cats}^{\mathsf{cocomp.}}, \quad (\perp_2)$$

witnessed by an adjoint equivalence of categories[2]

$$(\mathrm{Lan}_{\text{よ}} \dashv \text{よ}^*)\colon \quad \mathsf{Fun}^{\mathsf{cocont}}(\mathsf{PSh}(\mathcal{C}), \mathcal{D}) \underset{\text{よ}^*}{\overset{\mathrm{Lan}_{\text{よ}}}{\rightleftarrows}} \mathsf{Fun}(\mathcal{C}, \mathcal{D}), \quad (\perp)$$

natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{D} \in \mathrm{Obj}(\mathsf{Cats}^{\mathsf{cocomp.}})$, where

- We have a functor

$$よ_C^*\colon \mathsf{Fun}^{\mathsf{cocont}}(\mathsf{PSh}(C), \mathcal{D}) \longrightarrow \mathsf{Fun}(C, \mathcal{D})$$

defined by

$$よ_C^*(F) \overset{\mathrm{def}}{=} F \circ よ_C,$$

i.e. by sending a functor $F\colon \mathsf{PSh}(C) \longrightarrow \mathcal{D}$ to the composition

$$C \overset{よ_C}{\hookrightarrow} \mathsf{PSh}(C) \xrightarrow{F} \mathcal{D};$$

- We have a natural map

$$\mathsf{Lan}_{よ_C}\colon \mathsf{Fun}(C, \mathcal{D}) \longrightarrow \mathsf{Fun}^{\mathsf{cocont}}(\mathsf{PSh}(C), \mathcal{D})$$

computed on objects by

$$\begin{aligned}\left[\mathsf{Lan}_{よ_C}(F)\right](\mathcal{F}) &\cong \int^{A \in \mathcal{D}} \mathsf{Nat}(h_A, \mathcal{F}) \odot F_A\\ &\cong \int^{A \in \mathcal{D}} \mathcal{F}^A \odot F_A\end{aligned}$$

for each $\mathcal{F} \in \mathsf{Obj}(\mathsf{PSh}(C))$.

---

[1] In other words, the Yoneda embedding is indeed an embedding.

[2] In this sense, $\mathsf{PSh}(C)$ is the free cocompletion of C (although the term "cocompletion" is slightly misleading, as $\mathsf{PSh}(\mathsf{PSh}(C)) \overset{\mathrm{eq.}}{\ncong} \mathsf{PSh}(C)$).

Proof 7.3.3 ▶ Proof of Proposition 7.3.2

Item 1: Fully Faithfulness

Let $A, B \in \mathsf{Obj}(C)$. Applying Theorem 7.2.4 to the functor h_B (i.e. in the case $\mathcal{F} = h_B$), we have

$$\mathsf{Hom}_C(A, B) \cong \mathsf{Nat}(h_A, h_B).$$

Thus よ is fully faithful.

Item 2: Preservation and Reflection of Isomorphisms

This follows from Item 1 and Proposition 2.1.7.

Item 3: Uniqueness of Representing Objects Up to Isomorphism

By composing the isomorphisms $h_A \cong \mathcal{F} \cong h_B$, we get a natural isomorphism

$\alpha\colon h_A \overset{\cong}{\Longrightarrow} h_B$. By Item 2, we have $A \cong B$.

Item 4: As a Free Cocompletion: The Universal Property

This is a rephrasing of Item 5.

Item 5: As a Free Cocompletion: 2-Adjointness

See [nLab23c, Proposition 2.1].

7.4 Universal Objects

Definition 7.4.1 ▶ Universal Objects

The **universal object** associated to a representable functor $h_U\colon \mathcal{C} \longrightarrow \mathcal{D}$ is the element $u \in h_U(U)$ satisfying the following universal property:[1]

(UP) For each $B \in \mathrm{Obj}(\mathcal{C})$, the map

$$\begin{aligned} h_U(B) &\longrightarrow h_U(U) \\ (f\colon B \longrightarrow A) &\longmapsto h_U(f)(u) \end{aligned}$$

is a bijection.

[1] This is the element of $h_U(U)$ corresponding to the identity natural transformation $\mathrm{id}_{h_U}\colon h_U \Longrightarrow h_U$ under the isomorphism $h_U(U) \cong \mathrm{Hom}_{\mathsf{PSh}(\mathcal{C})}(h_U, h_U)$.

Remark 7.4.2 ▶ Why "Universal" Objects

In other words, a universal object u associated to a representable functor $h_U\colon \mathcal{C} \longrightarrow \mathcal{D}$ represented by U is universal in the sense that every element of $h_U(A)$ is equal to the image of u via $h_U(f)$ for a unique morphism $f\colon A \longrightarrow U$ of $\mathcal{C}$.

Example 7.4.3 ▶ Universal Numerable Principal G-Bundles

Let G be a group and consider the functor $\mathrm{Bun}^{\mathrm{num}}_G(-)\colon \mathrm{Ho}(\mathsf{Top})^{\mathsf{op}} \longrightarrow \mathsf{Sets}$ sending $[X] \in \mathrm{Ho}(\mathsf{Top})^{\mathsf{op}}$ to the set of numerable principal G-bundles on X. Then the universal numerable principal G-bundle $\gamma\colon \mathrm{E}G \longrightarrow \mathrm{B}G$ is a universal object for $\mathrm{Bun}^{\mathrm{num}}_G(-)$.

Furthermore, the map sending γ to a principal G-bundle $P \longrightarrow X$ on X is the pullback

$$f^*\colon \mathrm{Bun}_G^{\mathrm{num}}(\mathrm{BG}) \longrightarrow \mathrm{Bun}_G^{\mathrm{num}}(X)$$

of P along the homotopy class $[f]\colon X \longrightarrow \mathrm{BG}$ classifying P of maps $X \longrightarrow \mathrm{BG}$. See Algebraic Topology, ?? for more details.

8 The Contravariant Yoneda Lemma

8.1 Copresheaves

Let $\mathcal{C}$ be a category.

Definition 8.1.1 ▶ Copresheaves on a Category

A **copresheaf on** $\mathcal{C}$ is a functor $F\colon \mathcal{C} \longrightarrow \mathsf{Sets}$.

Definition 8.1.2 ▶ The Category of Copresheaves on a Category

The **category of copresheaves on** $\mathcal{C}$ is the category $\mathsf{CoPSh}(\mathcal{C})$ defined by

$$\mathsf{CoPSh}(\mathcal{C}) \stackrel{\mathrm{def}}{=} \mathsf{Fun}(\mathcal{C}, \mathsf{Sets}).$$

Remark 8.1.3 ▶ Unwinding Definition 8.1.2

In detail, the **category of copresheaves on** $\mathcal{C}$ is the category $\mathsf{CoPSh}(\mathcal{C})$ where

- *Objects.* The objects of $\mathsf{CoPSh}(\mathcal{C})$ are presheaves on $\mathcal{C}$;
- *Morphisms.* A morphism of $\mathsf{CoPSh}(\mathcal{C})$ from F to G is a natural transformation $\alpha\colon F \Longrightarrow G$;
- *Identities.* For each $F \in \mathrm{Obj}(\mathsf{CoPSh}(\mathcal{C}))$, the unit map

  $$\mathbb{1}_F^{\mathsf{CoPSh}(\mathcal{C})}\colon \mathrm{pt} \longrightarrow \mathrm{Nat}(F, F)$$

  of $\mathsf{CoPSh}(\mathcal{C})$ at F is defined by

  $$\mathrm{id}_F^{\mathsf{CoPSh}(\mathcal{C})} \stackrel{\mathrm{def}}{=} \mathrm{id}_F;$$

- *Composition.* For each $F, G, H \in \mathrm{Obj}(\mathsf{CoPSh}(\mathcal{C}))$, the composition map

  $$\circ_{F,G,H}^{\mathsf{CoPSh}(\mathcal{C})}\colon \mathrm{Nat}(G, H) \times \mathrm{Nat}(F, G) \longrightarrow \mathrm{Nat}(F, H)$$

of $\mathsf{CoPSh}(C)$ at (F, G, H) is defined by

$$\beta \circ^{\mathsf{CoPSh}(C)}_{F,G,H} \alpha \stackrel{\text{def}}{=} \beta \circ \alpha.$$

8.2 Corepresentable Copresheaves

Let C be a category, let $U, V \in \mathrm{Obj}(C)$, and let $f : U \longrightarrow V$ be a morphism of C.

Definition 8.2.1 ▶ The Corepresentable Copresheaf Associated to an Object

The **corepresentable copresheaf associated to** U is the copresheaf $h^U : C \longrightarrow \mathsf{Sets}$ on C where

- *Action on Objects.* For each $A \in \mathrm{Obj}(C)$, we have

$$h^U(A) \stackrel{\text{def}}{=} \mathrm{Hom}_C(U, A);$$

- *Action on Morphisms.* For each morphism $f : A \longrightarrow B$ of C, the image

$$h^U(f) : \underbrace{h^U(A)}_{\stackrel{\text{def}}{=}\mathrm{Hom}_C(U,A)} \longrightarrow \underbrace{h^U(B)}_{\stackrel{\text{def}}{=}\mathrm{Hom}_C(U,B)}$$

  of f by h^U is defined by

$$h^U(f) \stackrel{\text{def}}{=} f_*.$$

Definition 8.2.2 ▶ Corepresentable Copresheaves

A copresheaf $F : C \longrightarrow \mathsf{Sets}$ is **corepresentable** if $F \cong h^U$ for some $U \in \mathrm{Obj}(C)$.[1]

[1] In such a case, we call U a **corepresenting object** for F.

Definition 8.2.3 ▶ Corepresentable Natural Transformations

The **corepresentable natural transformation associated to** f is the natural transformation $h^f : h^V \Longrightarrow h^U$ consisting of the collection

$$\left\{ h^f_A : \underbrace{h^V(A)}_{\stackrel{\text{def}}{=} \mathrm{Hom}_C(V,A)} \longrightarrow \underbrace{h^U(A)}_{\stackrel{\text{def}}{=} \mathrm{Hom}_C(U,A)} \right\}_{A \in \mathrm{Obj}(C)}$$

where

$$h^f_A \stackrel{\text{def}}{=} f^*.$$

Theorem 8.2.4 ▶ The Contravariant Yoneda Lemma

Let $F : C \longrightarrow \mathsf{Sets}$ be a copresheaf on C. We have a bijection

$$\mathrm{Nat}\left(h^A, F\right) \cong F^A,$$

natural in $A \in \mathrm{Obj}(C)$, determining a natural isomorphism of functors

$$\mathrm{Nat}\left(h^{(-)}, F\right) \cong F.$$

Proof 8.2.5 ▶ Proof of Theorem 8.2.4

This is dual to Theorem 7.2.4. ☐

8.3 The Contravariant Yoneda Embedding

Definition 8.3.1 ▶ The Contravariant Yoneda Embedding

The **contravariant Yoneda embedding of** C is the functor[1]

$$Ɏ_C : C^{\mathrm{op}} \hookrightarrow \mathrm{Fun}(C, \mathsf{Sets})$$

where

- *Action on Objects.* For each $U \in \mathrm{Obj}(C)$, we have

$$Ɏ(U) \stackrel{\text{def}}{=} h^U;$$

- *Action on Morphisms.* For each morphism $f : U \longrightarrow V$ of $\mathcal{C}$, the image

$$\text{よ}(f) : \text{よ}(V) \longrightarrow \text{よ}(U)$$

of f by よ is defined by

$$\text{よ}(f) \overset{\text{def}}{=} h^f.$$

[1] *Further Notation:* Also written $h^{(-)}$, or simply よ.

PROPOSITION 8.3.2 ▶ PROPERTIES OF THE CONTRAVARIANT YONEDA EMBEDDING

Let $\mathcal{C}$ be a category.

1. *Fully Faithfulness.* The contravariant Yoneda embedding is fully faithful.[1]
2. *Preservation and Reflection of Isomorphisms.* Let $A, B \in \text{Obj}(\mathcal{C})$. The following conditions are equivalent:
   (a) We have $A \cong B$.
   (b) We have $h_A \cong h_B$.
   (c) We have $h^A \cong h^B$.
3. *Uniqueness of Representing Objects Up to Isomorphism.* Let $F : \mathcal{C} \longrightarrow \mathsf{Sets}$ be a copresheaf. If there exist objects A and B of $\mathcal{C}$ such that we have

$$\begin{aligned} h^A &\cong F, \\ h^B &\cong F, \end{aligned}$$

   then $A \cong B$.
4. *As a Free Completion: The Universal Property.* The pair $(\mathsf{CoPSh}(\mathcal{C})^{\mathsf{op}}, \text{よ})$ consisting of
   - The opposite $\mathsf{CoPSh}(\mathcal{C})^{\mathsf{op}}$ of the category of copresheaves on $\mathcal{C}$;
   - The contravariant Yoneda embedding $\text{よ} : \mathcal{C} \hookrightarrow \mathsf{CoPSh}(\mathcal{C})^{\mathsf{op}}$ of $\mathcal{C}$ into $\mathsf{CoPSh}(\mathcal{C})^{\mathsf{op}}$;

   satisfies the following universal property:

   (UP) Given another pair $(\mathcal{A}, F)$ consisting of
   - A complete category $\mathcal{A}$;
   - A continuous functor $F : \mathcal{C} \longrightarrow \mathcal{A}$;

there exists a continuous functor $\mathsf{CoPSh}(\mathcal{C})^{\mathsf{op}} \xrightarrow{\exists!} \mathcal{A}$, unique up to natural isomorphism, making the diagram

$$\begin{array}{ccc} & & \mathsf{CoPSh}(\mathcal{C})^{\mathsf{op}} \\ & \overset{よ}{\nearrow} \; \overset{\sim}{\Uparrow} & \downarrow{\scriptstyle \exists!} \\ \mathcal{C} & \xrightarrow[F]{} & \mathcal{A} \end{array}$$

commute, again up to natural isomorphism.

5. *As a Free Completion: 2-Adjointness.* We have a 2-adjunction

$$(\mathsf{CoPSh}^{\mathsf{op}} \dashv \iota)\colon \quad \mathsf{Cats} \underset{\iota}{\overset{\mathsf{CoPSh}^{\mathsf{op}}}{\underset{\perp_2}{\rightleftarrows}}} \mathsf{Cats}^{\mathsf{comp.}},$$

witnessed by an adjoint equivalence of categories

$$\left(\mathsf{Ran}^{\mathsf{op}}_{よ} \dashv よ^{*}\right)\colon \quad \mathsf{Fun}^{\mathsf{cont}}(\mathsf{CoPSh}(\mathcal{C})^{\mathsf{op}}, \mathcal{D}) \underset{よ^{*}}{\overset{\mathsf{Ran}^{\mathsf{op}}_{よ}}{\underset{\perp}{\rightleftarrows}}} \mathsf{Fun}(\mathcal{C}^{\mathsf{op}}, \mathcal{D}),$$

natural in $\mathcal{C} \in \mathsf{Obj}(\mathsf{Cats})$ and $\mathcal{D} \in \mathsf{Obj}(\mathsf{Cats}^{\mathsf{comp.}})$.

[1] In other words, the contravariant Yoneda embedding is indeed an embedding.

Proof 8.3.3 ▶ Proof of Proposition 8.3.2

This is dual to Proposition 7.3.2.

Appendices

A Miscellany

A.1 Concrete Categories

Definition A.1.1 ▶ Concrete Categories

A category $\mathcal{C}$ is **concrete** if there exists a faithful functor $F\colon \mathcal{C} \longrightarrow \mathsf{Sets}$.

A.2 Balanced Categories

Definition A.2.1 ▶ Balanced Categories

A category is **balanced** if every morphism which is both a monomorphism and an epimorphism is an isomorphism.

A.3 Monoid Actions on Objects of Categories

Let A be a monoid, let $\mathcal{C}$ be a category, and let $X \in \mathrm{Obj}(\mathcal{C})$.

Definition A.3.1 ▶ Monoid Actions on Objects of Categories

An **A-action on X** is a functor $\lambda\colon \mathsf{B}A \longrightarrow \mathcal{C}$ with $\lambda(\star) = X$.

Remark A.3.2 ▶ Unwinding Definition A.3.1

In detail, an **A-action on X** is an A-action on $\mathrm{End}_{\mathcal{C}}(X)$, consisting of a morphism

$$\lambda\colon A \longrightarrow \underbrace{\mathrm{End}_{\mathcal{C}}(X)}_{\overset{\text{def}}{=}\mathrm{Hom}_{\mathcal{C}}(X,X)}$$

satisfying the following conditions:

1. *Preservation of Identities.* We have

$$\lambda_{1_A} = \mathrm{id}_X.$$

2. *Preservation of Composition.* For each $a, b \in A$, we have

$$\lambda_b \circ \lambda_a = \lambda_{ab}, \qquad \begin{array}{ccc} X & \xrightarrow{\lambda_a} & X \\ & \searrow^{\lambda_{ab}} & \downarrow{\scriptstyle \lambda_b} \\ & & X. \end{array}$$

A.4 Group Actions on Objects of Categories

Let G be a group, let C be a category, and let $X \in \mathrm{Obj}(C)$.

Definition A.4.1 ▶ Group Actions on Objects of Categories

A G-**action on** X is a functor $\lambda\colon \mathrm{B}G \longrightarrow C$ with $\lambda(\star) = X$.

Remark A.4.2 ▶ Unwinding Definition A.4.1

In detail, a G-**action on** X is a G-action on $\mathrm{Aut}_C(X)$, consisting of a morphism

$$\lambda\colon G \longrightarrow \underbrace{\mathrm{End}_C(X)}_{\overset{\mathrm{def}}{=}\mathrm{Hom}_C(X,X)}$$

satisfying the following conditions:

1. *Preservation of Identities.* We have

$$\lambda_{1_A} = \mathrm{id}_X.$$

2. *Preservation of Composition.* For each $a, b \in A$, we have

$$\lambda_b \circ \lambda_a = \lambda_{ab}, \qquad \begin{array}{ccc} X & \xrightarrow{\lambda_a} & X \\ & \searrow^{\lambda_{ab}} & \downarrow{\scriptstyle \lambda_b} \\ & & X. \end{array}$$

B Miscellany on Presheaves

B.1 Limits and Colimits of Presheaves

Let $\mathcal{C}$ be a category.

Proposition B.1.1 ▶ Co/Limits of Presheaves are Computed Objectwise

Let $U \in \mathrm{Obj}(\mathcal{C})$. The functor

$$\begin{aligned} \mathsf{PSh}(\mathcal{C}) &\longrightarrow \mathsf{Sets} \\ \mathcal{F} &\longmapsto \mathcal{F}(U) \end{aligned}$$

commutes with limits and colimits: given a diagram $\mathcal{F}\colon \mathcal{I} \longrightarrow \mathsf{PSh}(\mathcal{C})$ of presheaves on $\mathcal{C}$, we have

$$\begin{aligned} \lim(\mathcal{F})_U &= \lim_{i\in\mathcal{I}}(\mathcal{F}_i(U)), \\ \operatorname{colim}(\mathcal{F})_U &= \operatorname*{colim}_{i\in\mathcal{I}}(\mathcal{F}_i(U)) \end{aligned}$$

for each $U \in \mathrm{Obj}(\mathcal{C})$.

Proof B.1.2 ▶ Proof of Proposition B.1.1

Omitted. ∎

B.2 Injective and Surjective Morphisms of Presheaves

Definition B.2.1 ▶ Injective and Surjective Morphisms of Presheaves

Let $\mathcal{C}$ be a category.

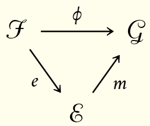
1. A map $\phi\colon \mathcal{F} \longrightarrow \mathcal{G}$ of presheaves is **injective** if for each $U \in \mathrm{Obj}(\mathcal{C})$, the map
$$\phi_U\colon \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$$
is injective.
2. A map $\phi\colon \mathcal{F} \longrightarrow \mathcal{G}$ of presheaves is **surjective** if for each $U \in \mathrm{Obj}(\mathcal{C})$, the map
$$\phi_U\colon \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$$

is surjective.

Proposition B.2.2 ▶ Monomorphisms and Epimorphisms of Presheaves

Let $\phi\colon \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of presheaves on C.

1. *Monomorphisms of Presheaves.* The following conditions are equivalent:
   (a) The morphism ϕ is a monomorphism in $\mathsf{PSh}(C)$.
   (b) The morphism ϕ is injective.
2. *Epimorphisms of Presheaves.* The following conditions are equivalent:
   (a) The morphism ϕ is an epimorphism in $\mathsf{PSh}(C)$.
   (b) The morphism ϕ is surjective.
3. *Isomorphisms of Presheaves.* The following conditions are equivalent:
   (a) The morphism ϕ is an isomorphism in $\mathsf{PSh}(C)$.
   (b) The morphism ϕ is injective and surjective.
4. *Epi-Mono Factorisation for Presheaves.* The morphism ϕ factors as an epimorphism followed by a monomorphism, i.e. there exists a factorisation of ϕ of the form



   with e an epimorphism and m a monomorphism.

Proof B.2.3 ▶ Proof of Proposition B.2.2

Item 1: Monomorphisms of Presheaves

We claim that Items (a) and (b) are indeed equivalent:[1]

- *Item (a)* $\Longrightarrow$ *Item (b)*. Suppose that ϕ is injective, and let $f, g\colon \mathcal{E} \rightrightarrows \mathcal{F}$ be two presheaf morphisms such that $\phi \circ f = \phi \circ g$. For each $U \in \mathrm{Obj}(C)$, we have

$$\phi_U \circ f_U = (\phi \circ f)_U = (\phi \circ g)_U = \phi_U \circ g_U.$$

Since ϕ is injective, so is ϕ_U. As injective morphisms are precisely the monomorphisms in Sets (Example 4.1.2), we have

$$f_U = g_U$$

for each $U \in \text{Obj}(C)$. Therefore $f = g$ and ϕ is a monomorphism.

- *Item (b)* $\Longrightarrow$ *Item (a)*. Conversely, suppose that ϕ is a monomorphism and let $U \in \text{Obj}(C)$ and $a, b \in \mathcal{F}(U)$ such that $\phi_U(a) = \phi_U(b)$. By the Yoneda lemma (Theorem 7.2.4), the sections a and b of $\mathcal{F}$ over U correspond to natural transformations

$$a' : h_U \Longrightarrow \mathcal{F},$$
$$b' : h_U \Longrightarrow \mathcal{F}.$$

Similarly, the sections $\phi_U(a)$ and $\phi_U(b)$ of $\mathcal{G}$ over U correspond to natural transformations

$$\phi \circ a' : h_U \Longrightarrow \mathcal{G}$$
$$\phi \circ b' : h_U \Longrightarrow \mathcal{G}.$$

As $\phi_U(a) = \phi_U(b)$, we have $\phi \circ a' = \phi \circ b'$, and hence $a' = b'$, as ϕ is a monomorphism. Therefore, $a = b$ and ϕ is injective.

Item 2: Epimorphisms of Presheaves

We claim that Items (a) and (b) are indeed equivalent:[2]

- *Item (a)* $\Longrightarrow$ *Item (b)*. Suppose that ϕ is surjective, and let $f, g : \mathcal{G} \rightrightarrows \mathcal{H}$ be two presheaf morphisms such that $f \circ \phi = g \circ \phi$. For each $U \in \text{Obj}(C)$, we have

$$f_U \circ \phi_U = (f \circ \phi)_U = (g \circ \phi)_U = g_U \circ \phi_U.$$

Since ϕ is surjective, so is ϕ_U. As surjective morphisms are precisely the epimorphisms in Sets (Example 5.1.2), we have

$$f_U = g_U$$

for each $U \in \text{Obj}(C)$. Therefore $f = g$ and ϕ is an epimorphism.

- *Item (b)* $\implies$ *Item (a)*. Conversely, suppose that ϕ is an epimorphism. Consider the presheaf $\mathcal{H} \colon \mathcal{C} \longrightarrow \mathsf{Sets}$ defined by

$$\mathcal{H}(U) = \mathcal{G}(U) \coprod_{\mathcal{F}(U)} \mathcal{G}(U)$$

for each $U \in \mathcal{C}$. Note that the action of $\mathcal{H}$ on morphisms is obtained by the functoriality of the pushout. By the definition of the pushout, we have

$$i_1 \circ \phi_U = i_2 \circ \phi_U,$$

which implies $i_1 = i_2$, since ϕ is an epimorphism. By Limits and Colimits, Lemma 3.5.2, ϕ is surjective.

Item 3: Isomorphisms of Presheaves

We claim that Items (a) and (b) are indeed equivalent:[3]

- *Item (a)* $\implies$ **??**. Suppose that ϕ is an isomorphism. Then so is $\phi_U \colon \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ for each $U \in \mathrm{Obj}(\mathcal{C})$. As isomorphisms in Sets are the maps that are both injective and surjective, ϕ_U is injective and surjective for each $U \in \mathrm{Obj}(\mathcal{C})$. Therefore ϕ is injective and surjective.

- *Item (b)* $\implies$ **??**. Conversely, suppose that ϕ is injective and surjective. Then so is ϕ_U for each $U \in \mathrm{Obj}(\mathcal{C})$. Furthermore, each ϕ_U is an isomorphism. This enables us to construct a natural transformation $\phi^{-1} \colon \mathcal{G} \longrightarrow \mathcal{F}$ consisting of the maps $\{\phi_U^{-1} \colon \mathcal{G}(U) \longrightarrow \mathcal{F}(U)\}$, which is an inverse to ϕ. Therefore ϕ is an isomorphism.

Item 4: Epi-Mono Factorisation for Presheaves

See [deJ20, Tag 00V9].

[1] *Reference:* [deJ20, Tag 00V7].
[2] *Reference:* [deJ20, Tag 00V7].
[3] *Reference:* [deJ20, Tag 00V7].

B.3 Subpresheaves

Let $\mathcal{C}$ be a category.

Definition B.3.1 ▶ Subpresheaves

A **subpresheaf** of a presheaf $\mathcal{G}$ on C is a subobject $\mathcal{F}$ of $\mathcal{G}$.

Remark B.3.2 ▶ Unwinding Definition B.3.1

In detail, a **subpresheaf** of $\mathcal{G}$ is an injective map $\mathcal{F} \hookrightarrow \mathcal{G}$ of presheaves, consisting therefore of a presheaf $\mathcal{F}$ satisfying the following conditions:

1. For each $U \in \mathrm{Obj}(C)$, we have $\mathcal{F}_U \subset \mathcal{G}_U$.
2. For each morphism $f : U \longrightarrow V$ of C, the diagram

$$\begin{array}{ccc} \mathcal{F}_U & \xrightarrow{\mathcal{F}_f} & \mathcal{F}_V \\ \big\downarrow & & \big\downarrow \\ \mathcal{G}_U & \xrightarrow[\mathcal{G}_f]{} & \mathcal{G}_V \end{array}$$

commutes.

B.4 The Image Presheaf

Let C be a category.

Definition B.4.1 ▶ Image Presheaves

The **image** of a morphism $\phi : \mathcal{F} \longrightarrow \mathcal{G}$ of presheaves on C is the presheaf $\mathrm{Im}(\phi)$ defined by

$$\mathrm{Im}(\phi)_U \stackrel{\text{def}}{=} \mathrm{Im}(\phi_U)$$

for each $U \in \mathrm{Obj}(C)$.

Proposition B.4.2 ▶ The Universal Property of the Image Presheaf

The image presheaf satisfies the following universal property:

(**UP**) There exists a unique injective morphism of presheaves $\mathrm{Im}(\phi) \xrightarrow{\exists!} \mathcal{G}$ such that the diagram

$$\begin{array}{ccc} & & \mathrm{Im}(\phi) \\ & \nearrow & \big\downarrow{\scriptstyle \exists!} \\ \mathcal{F} & \longrightarrow & \mathcal{G} \end{array}$$

commutes.

Proof B.4.3 ▶ Proof of Proposition B.4.2

Suppose we had a factorisation

$$\mathcal{F} \xrightarrow{\phi} \mathcal{G}' \hookrightarrow \mathcal{G},$$

with $\mathcal{G}'$ a subpresheaf of $\mathcal{G}$. Then we would have

$$\mathcal{F}(U) \xrightarrow{\phi_U} \mathcal{G}'(U) \hookrightarrow \mathcal{G}(U), \tag{B.4.1}$$

for each $U \in \mathrm{Obj}(\mathcal{C})$. But we know that in Sets the unique subset of $\mathcal{G}(U)$ giving the factorisation in Diagram (B.4.1) is $\mathrm{Im}(\phi_U)$. Thus $\mathcal{G}'(U) = \mathrm{Im}(\phi_U)$ for each $U \in \mathrm{Obj}(\mathcal{C})$ and $\mathcal{G}' = \mathrm{Im}(\phi)$. ∎

C Other Chapters

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169. Analytic Number Theory
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172. Elliptic Curves
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174. Automorphic Forms
175. Arakelov Geometry
176. Geometrisation of the Local Langlands Correspondence
177. Arithmetic Differential Geometry

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179. Constructions With Topological Spaces
180. Conditions on Topological Spaces
181. Sheaves on Topological Spaces
182. Topological Stacks
183. Locales
184. Metric Spaces

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185. Fibre Bundles, Vector Bundles, and Principal Bundles
186. Differential Forms, de Rham Cohomology, and Integration
187. Riemannian Geometry
188. Complex Geometry
189. Spin Geometry
190. Symplectic Geometry
191. Contact Geometry
192. Poisson Geometry
193. Orbifolds
194. Smooth Stacks
195. Diffeological Spaces

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196. Lie Groups
197. Lie Algebras
198. Kac–Moody Groups
199. Kac–Moody Algebras

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201. Spectral Sequences
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203. Operator K-Theory
204. Localisation and Completion of Spaces
205. Rational Homotopy Theory
206. p-Adic Homotopy Theory
207. Stable Homotopy Theory
208. Chromatic Homotopy Theory
209. Topological Modular Forms
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216. Finiteness Conditions on Morphisms of Schemes
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218. Varieties
219. Algebraic Vector Bundles
220. Divisors

Fundamental Groups of Schemes

221. The Étale Topology
222. The Étale Fundamental Group
223. Tannakian Fundamental Groups
224. Nori's Fundamental Group Scheme
225. Étale Homotopy of Schemes

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